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Asset Pricing Equilibria with Indivisible Goods

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Abstract

We study asset pricing of divisible assets based on consumption decisions of indivisible goods in a frictional market. Indivisibility matters for equilibria along with the trading mechanism. Bargaining generates a good's price that is not linked to the dividend value of the asset or the number of active buyers of the asset. In contrast, competitive search generates a price as a continuous function of the dividend and the number of buyers. In both cases, when the asset supply is scarce, the asset price bears a liquidity premium that closely relates to the dividend and the number of buyers. We also find that, for positive dividend values on the asset, unique stationary asset price equilibrium exists, while for negative dividend values, multiple equilibria occur. We show that lotteries are not used in any equilibria, but sellers are able to extract a positive surplus under bargaining with lotteries.

JEL: D51, G12.

Keywords: Asset, Competitive Search, Indivisibility, Lottery, Nash Bargaining.

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"Asset prices are also of fundamental importance for the macroeconomy because they provide crucial information for key economic decisions regarding physical investments and consumption. While prices of financial assets often seem to reflect fundamental values, history provides striking examples to the contrary, in events commonly labeled bubbles and crashes. Mispricing of assets may contribute to financial crises and, as the recent recession illustrates, such crises can damage the overall economy. Given the fundamental role of asset prices in many decisions, what can be said about their determinants?"
The Economic Sciences Prize Committee of the Royal Swedish Academy of Sciences (2013)

1 Introduction

Asset prices provide important information for economic decisions, and in particular, consumption decisions. Prices of financial assets sometimes reflect their fundamental values. However, historical examples of bubbles and crashes abound and mispricing of assets can create financial fragility and damaging effects on the overall macroeconomic aggregates. As exemplified by the 2013 Nobel Prize on "Understanding Asset Prices" allocated to Eugene Fama, Lars Peter Hansen, and Robert Shiller, it is of fundamental importance to understand the determinants of asset prices.

We provide a theory of asset prices based on consumption decisions. Recently, the environment of New Monetarism, based on the seminal work of Lagos and Wright (2005), provides conditions under which assets carry a liquidity premium. Assets, just like money, can convey liquidity for consumption purposes. In reality, assets can be used as a collateral for trade via credit in markets where there is a probability of default or imperfect credit, such as home equity loans, repo loans, and capital loans based on business valuation.[1] As argued in Lagos (2011) and Lagos et al. (2017), a collateralized loan is theoretically equivalent to having agents surrender the asset as a medium of exchange when credit is imperfect or impossible due to anonymity, lack of commitment, and punishment devices. When credit is imperfect, money is not the only object that can serve as the medium of

[1] See Kiyotaki and Moore (1997) for an early model.

exchange.

We study an economy with indivisible goods being traded with a divisible asset. In contrast to monetary models, the asset is in a fixed total supply and it bears an exogenous dividend, which can be positive or negative. We assume that buyers only want to consume one unit of the indivisible good but can hold any amount of assets.[2] While the case of divisible goods traded for assets has been studied, as in Wallace (2000), Geromichalos et al. (2007), Lagos and Rocheteau (2009), and Rocheteau and Wright (2013), the equilibrium consequences of indivisible goods traded for divisible assets have been neglected in the literature.[3]

We show how indivisible good and its pricing mechanism matter in the determinants of equilibrium asset prices, and how the predictions differ from those with divisible good. With indivisible goods, no adjustment can take place through the intensive margin and the total surplus from trade is fixed.

The equilibrium asset price emerging from the exchange process differs significantly depending on the trading mechanism used in the goods market. We consider an environment where the terms of trade in the goods market are determined by generalized Nash bargaining and by price posting with competitive search.[4] The former is an ex-post mechanism mapping the amount of asset carried in the market into outcomes in a bilateral trade,[5] while the latter is an ex-ante mechanism mapping the posted terms of trade into the choice of the amount of asset holding.

We find that under bargaining, buyers can commit to bringing the lowest amount of asset needed to make sellers indifferent between trading or not, and the bargained price does not depend on the dividend of the asset or buyers' participation in the good's market. The ex-post nature of the mechanism gives extra bargaining power to a buyer through the

[2] The perfectly divisible good (or discrete multi-units) is a convenient abstraction and a compelling assumption at the aggregate level. However, at the level of pairwise trades, there are many goods for which a typical buyer consumes a limited or even a unique amount.

[3] The notable exception is Han et al. (2016) who study indivisible goods traded in pure credit market and with fiat money.

[4] The competitive search framework we use is based on Moen (1997) and Mortensen and Wright (2002). For the use of competitive search in monetary models, see Rocheteau and Wright (2005) and Lagos and Rocheteau (2007). Unlike Rocheteau and Wright (2005), we do not consider price taking with search frictions. With indivisible goods, this gives rise to indeterminacy in price as in Jean et al. (2010) and Rabinovich (2017).

[5] See Gu and Wright (2016) for more details.

choice of the amount of asset carried for trade. The solution is akin to a take-it-or-leave-it offer by buyers, extracting the whole surplus from trade, independent of the bargaining power in the Nash bargaining problem. We show that with lotteries, i.e., the threat of not delivering the good, sellers are able to extract some of the surplus, even though lotteries end up not being used in equilibrium.

With price posting and competitive search, buyers choose their asset holdings after observing prices.[6] Hence, unlike bargaining, the price of the indivisible good is a continuous function of the participation rate of buyers and the cost of holding assets. The asset dividend also has an indirect effect on the good's price through participation.

An important implication of indivisibility is that in equilibrium, not all buyers choose to participate in the frictional good's market. This is in contrast to models where the good is perfectly divisible and all buyers always participate, as in Lagos and Rocheteau (2007). With divisible goods, sellers have two instruments to always assure non-negative surplus for buyers. With indivisible goods, only the price of the good can adjust. Thus, for a high cost of carrying the asset, if the equilibrium price is too high and all buyers participate, this can lead to a negative surplus for buyers from the congestion effect of the matching process. As a result, some buyers choose not to participate. Hence, the participation decision is tied to the terms of trade but also to the asset price and the dividend.

We find that if the dividend value is positive and high enough, all buyers participate in the decentralized market, and a unique asset price equilibrium exists under bargaining and competitive search. When the dividend is low, even negative, the matching congestion effect comes into play and reduces buyers' participation. The cost of carrying the asset is indirectly determined by the liquidity premium on the asset price, which itself, depends on the dividend value and the number of participating buyers. When the dividend is negative, in bargaining and competitive search, a higher number of active buyers implies a lower probability to trade and a lower cost of carrying the asset, which leads to the multiplicity of equilibrium in asset price.

[6] Alternatively, one can assume that the choice of asset holding is made prior to observing prices. Although it can be rationalized by a budgeting argument, especially when there is a cost of carrying liquidity, this seems to be a less natural assumption.

With bargaining, there are exactly two asset price equilibria: one with high participation by buyers and the other with low participation. Under price posting, the good's price is a continuous function of the dividend value and the number of active buyers, which leads to multiple asset price equilibria. We also show that the use of lotteries matters for participation in equilibrium under bargaining, but not under price posting and competitive search. Finally, we perform comparative statics under stationary equilibrium by varying the aggregate supply of the asset.

Related literature

Indivisibility of goods instead of assets matters and the consequences for equilibria differ from models with indivisible assets/money as in Shi (1995), Trejos and Wright (1995), and Wallace (2000).[7] Using the Shi-Trejos-Wright framework, Wallace (2000) considers indivisible assets traded for divisible goods with heterogeneity in positive asset dividends to obtain an equilibrium liquidity structure of asset yields. Duffie et al. (2005) (DGP) study Over-The-Counter (OTC) markets in a simplified Shi-Trejos-Wright environment with linear utility. There is a literature studying assets in fixed supply as a medium of exchange for consumption in the environment of New Monetarism with divisible goods and assets. Geromichalos et al. (2007) allow alternative assets to compete as media of exchange. They focus on the competition between fiat money and real financial assets and the equilibrium link between monetary policy and asset price. Assets are valued for their yield and liquidity services.

Lagos and Rocheteau (2009) study OTC market in a New Monetarism environment with heterogeneous demands for divisible assets. They model divisible assets and therefore remove the limited asset holding restriction of DGP. By modelling divisible assets Lagos and Rocheteau (2009) show that individual responses of asset demand constitute a fundamental feature of illiquid markets and are key determinants of trade volume, bid–ask spreads, and trading delays. Lagos (2011) studies an environment where assets and money are used as a medium of exchange for a divisible good. His focus is on the links between monetary policy and asset prices but he also studies the determinant of asset

[7] In Shi-Trejos-Wright (1995) and Wallace (2000), agents cannot accumulate more than one unit of money. See Julien et al. (2016), He and Wright (2017), and Wright, Kircher, Julien and Guerrieri (2017) for recent models based on Shi-Trejos-Wright using an asset instead of fiat money.

prices when assets are the only medium of exchange available in a frictional market with divisible goods. Rocheteau and Wright (2013) study liquidity and asset pricing in a New Monetarism model with asset being the medium of exchange. They generate multiple stationary equilibria, across which asset prices, market participation, capitalization, output, and welfare are positively related. They also generate a variety of nonstationary equilibria, including periodic, chaotic, and stochastic (sunspot) equilibria with recurrent market crashes.

These work relate to ours with the closest being Rocheteau and Wright (2013) focusing on divisible good and asset only use in trade. They endogenize participation by standard free entry of sellers in the decentralized market. Lagos and Rocheteau (2009) endogenize participation by allowing free entry of dealers in the OTC market. While we consider divisible assets as in those papers, we endogenize the participation of buyers via a minimum (non-zero) market utility condition. Unlike ours, these early papers do not consider competitive search in the decentralized market and they only focus on assets with positive real dividends. We study the equilibrium consequences of negative real dividends on the asset.[8] In the above papers, as in ours, asset prices carry a liquidity premium because they serve directly as media of exchange. In Geromichalos and Herrenbrueck (2016), asset prices carry a liquidity premium because agents sell them for cash in a secondary asset markets, and hence, asset liquidity is indirect.

The paper is organized as follows. In Section 2, we describe the environment. In Section 3, we consider an economy with asset used as the medium of exchange, prove existence and uniqueness of asset equilibrium when bargaining or competitive search serves as the trading mechanism, and study the effects of changing asset supply. In Section 4, we introduce lotteries, and Section 5 concludes.

[8]The economics and finance literature focuses essentially on assets with real positive dividends. However, in economies with positive inflation, stocks and bonds typically have positive nominal returns but not necessarily positive real returns. Checking accounts also bear negative real returns, especially in times of inflation. Traditionally, commodity money had storage costs and a risk of loss/theft that map into negative real returns.

2 Environment

The environment is based on the alternating markets framework of Rocheteau and Wright (2005).[9] Time is discrete and a continuum of buyers and sellers, with measures N and 1, live forever. Each period, $n \leq N$ buyers and all sellers participate in two consecutive markets. The first market to open is a frictional decentralized market (DM). In the DM, meetings occur according to a general meeting technology, which is assumed homogeneous of degree one. Given the buyer-seller ratio $n \leq N$, which is also the measure of participating buyers in the DM, the meeting rate for sellers and buyers are $\alpha(n)$ and $\alpha(n)/n$, respectively. Assume $\alpha' > 0$, $\alpha'' < 0$, $\alpha(0) = 0$, $\lim_{n\to\infty} \alpha(n) = 1$, and $\lim_{n\to 0} \alpha'(n) = 1$. The second market to open each period is a frictionless centralized market (CM). Agents discount between periods with factor $\beta \in (0, 1)$, but not across markets within a period, and $r = 1/\beta - 1$ is the discount rate.

Both buyers and sellers consume a divisible good in the CM, while only buyers consume and sellers produce an indivisible good in the DM. Buyers' preferences within a period are given by $U(x) - h + u\mathbf{1}$, where x is CM consumption, h is CM labor, u is DM utility from consuming the indivisible good, and $\mathbf{1}$ is an indicator function, giving 1 if trade occurs and 0 otherwise. Let x be the numeraire, and we assume that x is produced one-to-one from labor h. Sellers' preferences are $U(x) - h - c\mathbf{1}$ with DM good produced at constant cost $c < u$.

The only asset in the economy is a real asset, a. The real asset is perfectly divisible and recognizable and it cannot be counterfeited. The total asset supply is fixed at A^s and at $t = 0$ each buyer is endowed with A^s/N assets. In subsequent periods the asset is traded competitively in the CM at price φ. The real asset generates an exogenously determined dividend ρ, paid in the CM in terms of x. The dividend value can be either positive or negative.

Trade in the DM implies a price and quantity bundle $(p, q) \in \mathbf{P} \times \mathbf{Q}$ where $\mathbf{P} = \{0 \leq p \leq L\}$ and $\mathbf{Q} = \{0, 1\}$. We use L to denote the total available liquidity in the economy,

[9] The original alternating markets framework by Lagos and Wright (2005) has agents receiving a preference shock in the CM revealing whether they will be a buyer or a seller in the DM. In Rocheteau and Wright (2005), buyers are always buyers and sellers are always sellers. All our results hold for both frameworks.

with $L = (\varphi + \rho)a$ being the value of asset holdings of a buyer in the DM. More specifically, in any period t, buyers pay $\varphi_t a$ to aquire the asset in the CM. Buyers can use the asset as payment for the indivisible good in the subsequent DM, as long as $p \le \left(\varphi_{t+1} + \rho\right) a$, since the anticipated return of the asset at the beginning of period $t+1$ is $\left(\varphi_{t+1} + \rho\right) a$. After successful trade, sellers acquire the real assets which they can sell in the subsequent CM at price φ_{t+1}. A necessary condition for sellers to hold assets is $\varphi_t \le \beta(\varphi_{t+1} + \rho)$. In a steady state equilibrium the equation reduces to $\varphi \le \frac{\rho}{r} \equiv \varphi^F$ where φ^F represents the asset price when the asset is priced at its fundamental value. Therefore, sellers will only hold assets as store of value when they are fundamentally priced.

Define $s_t = \varphi_t / \beta(\varphi_{t+1} + \rho) - 1$ as the spread on the real asset.[10] Alternatively, the spread can be considered as the liquidity premium of holding the asset. Given discounting, the anticipated buyer's total cost (gain) of carrying the asset for trade in the DM is

$$s_t \beta \left(\varphi_{t+1} + \rho\right) a = \left[\varphi_t - \beta\left(\varphi_{t+1} + \rho\right)\right] a > 0 \ (< 0), \tag{1}$$

When $s_t = 0$ the asset is priced fundamentally and does not include a liquidity premium. If $s_t < 0$, both buyers and sellers would demand the asset for store of value since they can make a capital gain by holding the asset. Given the fixed asset supply A^s, this would lead to excess demand and hence an increase in φ_t until it hits $s_t = 0$. If there is still an excess demand when the asset is priced at the fundamental value, we have $s_t > 0$. In that case, sellers no longer hold the asset, and buyers hold all the assets for transaction purposes, paying a liquidity premium for acquiring the asset. Therefore, in any equilibrium path, including the stationary equilibrium, we have $s_t \ge 0$. When the asset is costly to carry, buyers have no incentives to bring assets with the total value being more than p. Therefore, under any trading mechanism, the feasibility constraint $p = (\varphi_{t+1} + \rho)a$ is always binding.

Once the value of the spread, s_t, is known[11], given β, ρ, and the anticipated φ_{t+1}, the

[10] Using $\beta = 1/(1+r)$, rewrite the spread equation as $1 + s_t = (1+r)\varphi_t/(\varphi_{t+1} + \rho)$. This is reminescent of the Fisher equation used in monetary models where $1 + i = (1+r)\phi_t/\phi_{t+1}$ with i being the nominal interest rate set by monetary policy and ϕ the price of money in terms of the numeraire goods in the CM. With money i is exogenous while with assets the spread s is endogenous. Thus, i is the spread on money.

[11] As will be demonstrated in the next section, the supply and demand of liquidity are both functions

asset price φ_t can be derived. For any given value of s_t, in any equilibrium, from (1), the asset price at time t is given by

$$\varphi_t = \beta(\varphi_{t+1} + \rho)(1 + s_t). \tag{2}$$

3 The Model

Assume that agents in the DM cannot commit and there are no enforcement or punishment mechanisms. Hence, buyers must bring a medium of exchange into the DM to pay sellers. Let real assets be that medium of exchange.[12]

Let $W_t(a)$ and $V_t(a)$ denote the value functions of an agent holding a units of the real asset when entering the CM and DM, respectively. Buyers in the CM obtain:

$$W_t^b(a) = \max_{x,h,\hat{a}} \left\{ U(x) - h + \beta V_{t+1}^b(\hat{a}) \right\} \text{ s.t. } x = (\varphi_t + \rho)\, a + h - \varphi_t \hat{a},$$

where $\hat{a}$ is the asset holding carried into the following DM. Buyers participate in the DM if $V_{t+1}^b \geq 0$. Eliminating h from the budget constraint and solving for optimal x^* yields,

$$W_t^b(a) = \Sigma + (\varphi_t + \rho)\, a + \max_{\hat{a}} \left\{ \beta V_{t+1}^b(\hat{a}) - \varphi_t \hat{a} \right\}, \tag{3}$$

where $\Sigma = U(x^*) - x^*$ and $U'(x^*) = 1$. Similarly, for a seller with a we have

$$W_t^s(a) = \Sigma + (\varphi_t + \rho)\, a + \max_{\hat{a}} \left\{ \beta V_{t+1}^s(\hat{a}) - \varphi_t \hat{a} \right\}. \tag{4}$$

The buyer's payoff in the DM is

$$V_t^b(a) = \frac{\alpha(n)}{n} \left[u + W_t^b\left(a - \frac{p}{\varphi_t + \rho} \right) \right] + \left[1 - \frac{\alpha(n)}{n} \right] W_t^b(a), \tag{5}$$

where p is the price paid by the buyer for the DM good, measured by the numeraire.

of s, which will allow us to determine s.

[12] Alternatively, one can assume that agents use assets as collateral to get credit in the DM, as in Kiyotaki and Moore (1997) with the pledgeability parameter to be 1, and all the results still hold. Lagos et al. (2017) have elaborated on these two setups being mathematically equivalent.

Using $\partial W_t^b / \partial a = \varphi_t + \rho$,

$$V_t^b(a) = \frac{\alpha(n)}{n}(u - p) + W_t^b(a). \tag{6}$$

Similarly, for sellers, we have

$$V_t^s(a) = \alpha(n)(p - c) + W_t^s(a). \tag{7}$$

As mentioned above, sellers will only hold assets in the DM for store of value purposes, and only if the asset is priced fundamentally.

In the next two subsections we analyze the implications of having an indivisible DM good traded with real assets both on DM prices and asset prices under generalized Nash bargaining and price posting with directed search.

3.1 Bargaining

In this section, we consider the case that the DM price is determined by generalized Nash bargaining. Buyers and sellers face the following bargaining problem in the DM,

$$\max_p (u - p)^\eta (p - c)^{1-\eta} \text{ s.t. } p \leq \left(\varphi_{t+1} + \rho\right) a,\ u - p \geq 0,\ p - c \geq 0. \tag{8}$$

Definition 1 *A stationary symmetric bargaining equilibrium, given s, is an allocation $\{p^b, n, V^b, V^s\}$ such that*

(i) the equilibrium price p^b solves (8);

(ii) sellers' participation: $V^s \geq W^s$;

(iii) buyers' participation: $V^b \geq W^b$, for all $n \leq N$.

The feasibility constraint $p = (\varphi_{t+1} + \rho)a$ is binding and hence $c \leq (\varphi_{t+1} + \rho)a \leq \bar{p}^B$, where $\bar{p}^B = (1 - \eta)u + \eta c$ is the unconstrained bargaining solution. Any negotiated price $p \in [c, \bar{p}^B]$ is a potential bargaining solution.[13] Substituting V_{t+1}^b into W_t^b and a buyer's

[13] Using proportional bargaining, i.e., Kalai and Smorodinsky (1975), for terms of trade yields the same results.

CM value function is:

$$W_t^b(a) = \Sigma + (\varphi_t + \rho)\, a + \beta W_{t+1}^b(0) + \max_{\hat{a}} \left\{ \beta \frac{\alpha(n)}{n} (u - p) + \left[\beta\left(\varphi_{t+1} + \rho\right) - \varphi_t\right] \hat{a} \right\}.$$

The buyer's problem can be rewritten as

$$\tilde{V}^b(n, s_t) \equiv \max_{\hat{a} \in [\underline{a}, \bar{a}]} \beta \left\{ \frac{\alpha(n)}{n} \left[u - \left(\varphi_{t+1} + \rho\right) \hat{a}\right] - s_t \left(\varphi_{t+1} + \rho\right) \hat{a} \right\}, \tag{9}$$

where $\underline{a} = c/(\varphi_{t+1}+\rho)$ and $\bar{a} = [(1-\eta)u+\eta c]/(\varphi_{t+1}+\rho)$. It is apparent that $\tilde{V}^b(\hat{a})$ is strictly decreasing in $\hat{a}$ for all values of $s_t \geq 0$. The optimal solution satisfies $\hat{a}^*(\varphi_{t+1} + \rho) = c$. With bargaining, a buyer can commit to not paying more than the seller's reservation price c. This result is equivalent to the monetary equilibrium in Han et al (2016). They demonstrate that when buyers bring real monetary balances into the DM the bargained price reduces to $p = c$. Therefore, under bargaining and indivisible DM goods, when a buyer brings a medium of exchange into the DM, he is able to extract the full surplus from trade.

The buyer's expected net value from participating in the DM is

$$\tilde{V}^b(n, s_t) = \beta \left[\frac{\alpha(n)}{n} (u - c) - s_t c \right] \geq 0, \tag{10}$$

which is the discounted expected total benefit net of the cost of carrying the asset. The measure of DM buyers $n^* \leq N$ is determined by a free entry condition, $\tilde{V}^b(n^*, s_t) \geq 0$. More specifically, $\tilde{V}^b(n^*, s_t) = 0$ if $n^* < N$ and $\tilde{V}^b(n^*, s_t) > 0$ if $n^* = N$. When $s_t = 0$, the asset is priced at its fundamental value and $\tilde{V}^b(n^*, s_t) > 0$, $\forall n^* \leq N$. Hence, all buyers participate in the DM. However, a higher s_t starts to have an effect on the number of buyers participating in the DM when $\tilde{V}^b(n^*, s_t) = 0$. Since $\alpha(n)/n$ is strictly decreasing in n (a congestion effect for buyers), a larger s_t implies a lower n^* when $\tilde{V}^b(n^*, s_t) = 0$. This leads to $n^*(s_t)$ with $\partial n^*/\partial s_t < 0$.

Asset prices in a stationary equilibria satisfy $\varphi_t = \varphi$ and $s_t = s$, $\forall t$. To establish equilibrium existence and uniqueness, we start by defining the demand, $L^d(s)$, and supply, $L^s(s)$, for liquidity. Under bargaining, they are $L^d(s) = n^*(s)c$ and $L^s(s) = (\varphi(s) + \rho)A^s$, where $n^*(s)$ is determined from $\tilde{V}^b(n^*, s_t) = 0$ and $\varphi(s)$ is solved from (2),

$$\varphi(s) = \frac{\beta\rho(1+s)}{1-\beta(1+s)} = \frac{\rho(1+s)}{r-s}. \tag{11}$$

When $s = 0$,

$$\varphi = \rho/r \equiv \varphi^F \tag{12}$$

is the fundamental asset price in a stationary equilibrium. If $s \neq r$, we have

$$L^d(s) = n^*(s)\,c = \frac{\rho\,(1+r)}{r-s}A^s = L^s(s), \tag{13}$$

and when $r = s$, the supply of liquidity is infinitely elastic. The stationary equilibrium $s^* \geq 0$ depends on critical values of parameters (ρ, r, c, A^s). From (13), we can solve for

$$s^* = r - \frac{\rho A^s(1+r)}{n^* c}, \tag{14}$$

where n^* is determined by the free entry condition (10).

When $s > 0$, inserting (14) into (11) yields the asset price $\varphi^{n^*} = \frac{n^*c}{A^s} - \rho$. Evaluating at $n^* = N$:

$$\varphi^N = \frac{Nc}{A^s} - \rho > \varphi^F \tag{15}$$

and the asset price is strictly decreasing in ρ and A^s.

Define ρ^N to be the cutoff dividend value for which every buyer is willing to enter the DM, and s^N denotes the liquidity premium associated with ρ^N. Let $\underline{\rho}$ and $\bar{s}^B$ be the cutoff for the existence of DM trades. Furthermore, let ρ^F denote the cutoff value above which assets are priced fundamentally, corresponding with $s = 0$. The following two lemmas characterize the aggregate demand and supply for liquidity:

Lemma 1 *There exist $\bar{s}^B \geq r$ and $s^N \leq \bar{s}^B$, such that: (i) for $s \leq s^N$, $\exists!\ L^d$ with $n^* = N$ and $L^d = Nc$; (ii) for $s \in (s^N, \bar{s}^B]$, $\exists!\ L^d$ with $n^* < N$, $L^d = n^*c$ and $dL^d/ds < 0$; (iii) for $s > \bar{s}^B$, $\nexists\ n^* > 0$ and L^d is not well-defined.*

Next lemma characterizes the aggregate supply of liquidity.

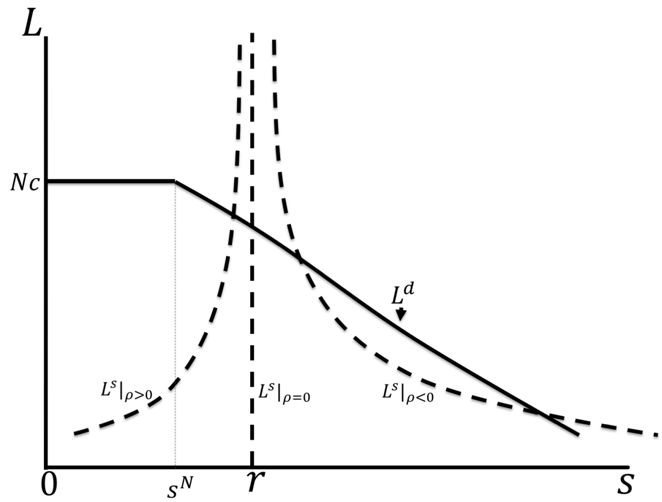

Figure 1: Supply and Demand for Liquidity under Bargaining

Lemma 2 *(i) For $\rho < 0$ $(r < s)$, L^s is convex and $dL^s/ds < 0$; (ii) for $\rho = 0$, L^s is perfectly elastic at $s = r$; (iii) for $\rho \in (0, \rho^F)$ $(0 < s < r)$, L^s is concave and $dL^s/ds > 0$; (iv) for $\rho \geq \rho^F$, L^s is perfectly elastic at $s = 0$.*

The two lemmas are depicted visually in Figure 1. In equilibrium, dividend values $\rho \geq 0$ correspond with a unique value of the spread, s. However, negative dividend values can generate multiple values for s. Since the asset price is a function of s, as demonstrated in (11), this results in multiple asset pricing equilbria for $\rho < 0$, as is summarized in Proposition 1.

We summarize equilibria in the following proposition:

Proposition 1 *In the model with bargaining: (i) for $\rho \geq \rho^F > 0$, $\exists!$ stationary equilibrium (SE) with $\varphi = \varphi^F$ and $n^* = N$; (ii) for $\rho \in [\rho^N, \rho^F)$ and $\rho^N > 0$, $\exists!$ SE with $\varphi = \varphi^N > \varphi^F$ and $n^* = N$; (iii) for $\rho \in [0, \rho^N]$ and $\rho^N > 0$, $\exists!$ SE with $\varphi = \varphi^{n^*} > \varphi^F$ and $n^* < N$. (iv) for $\rho \in [\rho^N, 0)$ and $\rho^N < 0$, $\exists$ two SE, one with $\varphi = \varphi^N > \varphi^F$ and $n^* = N$; and the other with $\varphi = \varphi^{n^*} > \varphi^F$ but $\varphi^{n^*} < \varphi^N$ and $n^* < N$; (v) for $\rho \in (\underline{\rho}, \rho^N)$ and $\rho < 0$, $\exists$ two SE, with $\varphi_h = \varphi^{n_h^*} > \varphi_l = \varphi^{n_l^*} > \varphi^F$ and $n_\ell^* < n_h^* < N$; (vi) for $\rho = \underline{\rho} < 0$, $\exists!$ SE with $\varphi = \varphi^{n^*} > \varphi^F$ and $n^* < N$; (vii) for $\rho < \underline{\rho}$, $\nexists$ SE with an active DM.*

To summarize Proposition 1, there exists a unique stationary equilibrium for all positive dividend values, while for negative dividend values there are two equilibria.

For case (i), if $\rho \geq \rho^F > 0$, where ρ^F solves (13) with $n^* = N$, a buyer does not need to carry many assets for trading purposes in the DM. The marginal holder of assets is a seller. Assets are priced at the fundamental value $\varphi^F = \rho/r$. In this case, we have $s = 0$, and carrying assets is costless. The participation constraint (10) becomes $\tilde{V}^b(n) = \beta(u-c)\alpha(n)/n \geq 0$, which is positive for all $n \leq N$. All buyers participate in the DM and the liquidity need of all buyers is satisfied.

For case (ii), if $\rho \in [\rho^N, \rho^F) \subset \mathbb{R}_+$, $s^* > 0$ and from (10) and (14), the participation constraint becomes

$$\tilde{V}^b(n^*) = \beta\frac{\alpha(n^*)}{n^*}(u-c) - (1-\beta)c + \frac{\rho A^s}{n^*} \geq 0.$$

Define $B(n^*) = \beta(u-c)\alpha(n^*)/n^* - (1-\beta)c$ as the buyer's discounted total benefit minus the flow payment in a stationary equilibrium. Then, we can write the per-buyer participation value in the DM as

$$\tilde{V}^b(n^*) = B(n^*) + \frac{\rho A^s}{n^*}.$$

Since $B(n^*)$ is strictly decreasing in n^*, for all $\rho \geq 0$, $\tilde{V}^b(n^*)$ is also strictly decreasing in n^*. Buyers receive an extra benefit from carrying the asset with positive dividend. However, it can be that $B(n^*) < 0$, and buyers still participate given the positive benefit of carrying the asset with a positive dividend. We define ρ^N by $\tilde{V}^b(N) = 0$. For all $\rho \geq \rho^N$, we have $\tilde{V}^b(n^*) > 0$ for all $n^* \leq N$, and thus $n^* = N$ with all buyers participating. The marginal holder of assets is a buyer, who cares about liquidity. The buyer's liquidity demand drives up the asset price to be above its fundamental value, and sellers no longer hold assets.

For case (iii) $\rho \in [0, \rho^N)$, we have $s^* > 0$ and $\tilde{V}^b(N) < 0$. If all buyers were to participate, the congestion effect due to the properties of the matching technology is too strong to generate a positive surplus. Hence, the unique equilibrium entails $n^* < N$ and $\varphi^{n^*} = n^*c/A^s - \rho$. When $\rho = 0$, assets are equivalent to money with a constant supply, and the results are comparable to Han et al. (2016). The above cases are illustrated in Figure 2.

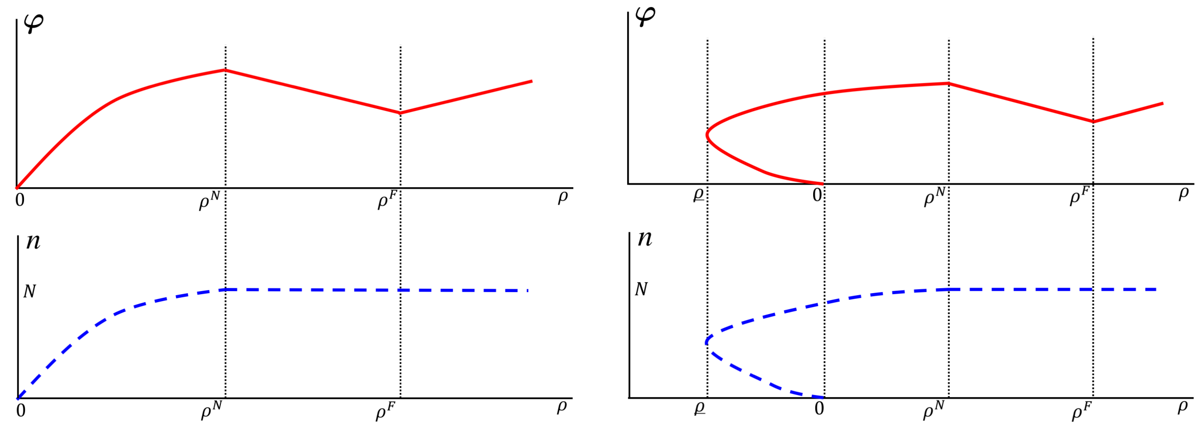

Figure 2: Bargaining Equilibrium, Cases (i) - (iii)

When $\rho < 0$, there exist two stationary equilibria up to a cutoff value $\underline{\rho} < 0$, at which the stationary equilibrium is unique again, as in case (vi). The cases for which $\rho^N < 0$ are illustrated in Figure 3.

From the participation constraint, we can show that there is a unique $\underline{\rho}$. The cutoff dividend value is such that, while entry maximizes a buyer's benefit from entering the DM, the cost of carrying liquidity, $\underline{\rho}A^s < 0$, leaves zero net surplus. Interestingly, the equilibrium level of participation corresponding to $\underline{\rho}$ is characterized by

$$\alpha'(n^*) = \frac{(1-\beta)c}{\beta(u-c)},$$

which is unique and relates to the well-known Hosios (1990) condition for efficient entry.[14] The marginal contribution to the matching process by a buyer equals the flow cost of entry, or equivalently, the buyer's share, $(1-\beta)c$, over the discounted total surplus, $\beta(u-c)$. Therefore, the endogenous entry of buyers is constrained efficient. For all $\rho < \underline{\rho} < 0$, there is no equilibrium, which is case (vii).

For other values of $\rho \in [\rho^N, 0)$ and $\rho \in (\underline{\rho}, \rho^N)$, as in cases (iv) and (v), there exist two different levels of buyer's participation in the DM, high and low, corresponding to two asset prices. When $\rho < 0$, the negative dividend is similar to a storage cost, akin to

[14]The Hosios' entry condition is most commonly expressed as $\varepsilon(n) = \frac{(1-\beta)c}{\beta(u-c)\alpha(n)/n}$ with $\varepsilon(n) = \alpha\prime(n)n/\alpha(n)$.

Kiyotaki and Wright (1989). In the first region of ρ, the equilibria entail one with full participation $n_h^* = N$ and $\varphi^N > \varphi^F$, and the other with $n_\ell^* < N$ and $\varphi^N > \varphi^{n_\ell^*} > \varphi^F$. In the second region $(\underline{\rho}, \rho^N)$, there is no full participation and the asset price is lower in the equilibrium with less entry, i.e., $\varphi^{n_\ell^*} < \varphi^{n_h^*}$ with $n_\ell^* < n_h^* < N$.

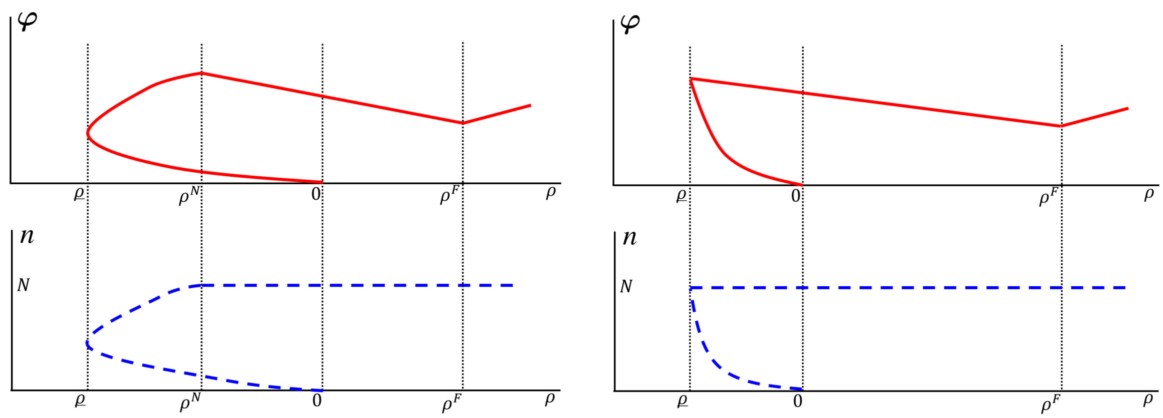

Figure 3: Bargaining Equilibrium, $\rho^N < 0$

To understand the intuition, note that from (11), the asset price is negatively related to the spread s when $\rho < 0$. When the equilibrium participation is low, a smaller liquidity demand drives down the asset price φ, which implies a larger spread s. As such, a low asset price implies a high spread, making participation more costly. The congestion effect in the matching process leads to lower participation. Buyers now face a high probability of trade in the DM, but they face a large cost of holding assets. This transition continues until free entry condition is restored. This is easily represented by

$$\tilde{V}^b(n_\ell^*) = B(n_\ell^*) + \frac{\rho A^s}{n_\ell^*} = B(n_h^*) + \frac{\rho A^s}{n_h^*} = \tilde{V}^b(n_h^*) = 0,$$

with $B(n_\ell^*) > B(n_h^*)$ and $\rho A^s/n_\ell^* > \rho A^s/n_h^*$ since $\rho < 0$. When $\rho < 0$, there is a coordination issue among buyers at the participation stage, weighing the congestion effect of the matching process with the cost of carrying liquidity. Yet, unless the negative dividend value hits $\underline{\rho}$, the Hosios' entry condition fails. It is as if the cutoff value removes the coordination problem and generates uniqueness. Note that when $\rho \geq 0$, this coordination problem does not exist, since $\partial\varphi/\partial s \geq 0$. Less participation implies a higher probability to trade and a lower spread of holding assets, and hence the equilibrium is unique.

A similar trade-off, between the probability of trade and the trading price, generates multiple equilibria in Rocheteau and Wright (2005). In their monetary model, the cost of holding liquidity is exogenously given at i, the nominal interest rate, while s is endogenously determined by the demand and supply of assets in our model. In a monetary economy, a lower participation in the DM implies a higher probability of trade for buyers. Since the cost of holding money is not adjusting with n, buyers are strictly better off, and the above coordination issue does not exist.

At this point, one may wonder what happens to the stationary equilibrium if we change the total asset supply A^s. To understand the effects, we define $\mathbf{A} = \rho A^s$ to be the amount of total liquidity in the market. Note that the asset price and the buyers' participation are affected by both the dividend value ρ and the supply of assets A^s. If $\rho = 0$, we are back to the money case and intuitively, $\partial\varphi/\partial A^s < 0$ and $\partial n/\partial A^s = 0$. More interesting cases are when $\rho \neq 0$. In particular, when $\rho > 0$, an increase in either ρ or A^s will make liquidity $\mathbf{A}$ more abundant, but if $\rho < 0$, i.e., assets bear a storage cost, an increase in ρ or a decrease in A^s actually implies liquidity being less costly. The following proposition summarizes the effects of changing $\mathbf{A}$ under bargaining, and w.l.o.g. we hold the value of A^s constant.

Proposition 2 *In the unique SE with bargaining ($\rho > 0$) and the SE with high participation ($\rho < 0$): (i) for $\mathbf{A} \geq \rho^F A^s$, we have $\partial\varphi/\partial\mathbf{A} > 0$ and $\partial n/\partial\mathbf{A} = 0$; (ii) for $\mathbf{A} \in [\rho^N A^s, \rho^F A^s)$, $\partial\varphi/\partial\mathbf{A} < 0$ and $\partial n/\partial\mathbf{A} = 0$; (iii) for $\mathbf{A} \in (\underline{\rho} A^s, \rho^N A^s)$, $\partial\varphi/\partial\mathbf{A} > 0$ and $\partial n/\partial\mathbf{A} > 0$. In the SE with low participation ($\rho < 0$), for $\mathbf{A} \in (\underline{\rho} A^s, 0)$, $\partial\varphi/\partial\mathbf{A} < 0$ and $\partial n/\partial\mathbf{A} < 0$.*

When $\mathbf{A}$ is large, i.e., liquidity is abundant in the economy, all transactional needs of buyers are satisfied, and hence assets have no liquidity premium and are priced at the fundamental value. All buyers participate in the DM due to zero holdup cost.

When liquidity is relatively scarce, i.e., case (ii) of Proposition 2 and $\mathbf{A}$ is not very large, buyers start to pay a liquidity premium for holding assets. While all the buyers still participate in the DM, the asset price increases with a drop in $\mathbf{A}$. For simplicity, we fix the value of A^s and a drop in $\mathbf{A}$ implies a decrease in ρ. Alternatively, a decrease in asset supply A^s when $\rho > 0$ or an increase in A^s when $\rho < 0$ may also cause a shortage of liquidity. In order to satisfy the demand of assets for transaction purposes in the DM, asset price actually increases facing the shortage of liquidity, which implies an even higher liquidity premium. This result echoes a key point in the New Monetarist literature: liquidity plays a key role in determining the price of an asset.

As $\mathbf{A}$ decreases further, liquidity becomes even more scarce, and buyers start to drop out of the DM. An adjustment in the extensive margin leads to two opposite effects. As ρ decreases, the asset price also drops and more buyers choose not to enter the DM, while the participating buyers are compensated by a higher probability to trade. This is the standard “hot potato” effect: people trade faster as the cost of transaction increases. On the other hand, when $\rho < 0$, the asset is toxic, i.e., it has a storage cost. If a decrease in $\mathbf{A}$ is caused by more toxic assets in the economy, there is an incentive for more buyers to participate in the DM. More buyers lead to a higher demand for assets in DM transactions, which drives up the liquidity premium, offsets the negative dividend, increases the asset price, and lowers the cost of carrying liquidity. This channel works as if more buyers get involved to share the toxic nature of the assets, and we name it the “poison apple” effect. Both effects are present when $\rho < 0$. While the “hot potato” effect dominates in the equilibrium with a high entry, the “poison apple” effect prevails in the equilibrium with a low entry.

Note that we need both adjustable extensive margin and endogenous liquidity cost to have the “poison apple” effect. Lagos and Rocheteau (2005) generates the “hot potato” effect but not the second one, since agents do not have a participation decision. Liu et al. (2011) study the “hot potato” effect through the extensive margin, but the cost of carrying liquidity is exogenous in their model. In fact, the “poison apple” effect does not exist in any monetary models, because the cost of holding money is always exogenous and independent of participation.

3.2 Competitive Search

In this section, we study competitive search equilibrium with price posting.[15] As in Moen (1997) and Rocheteau and Wright (2005), instead of a single DM, there exist a continuum of submarkets, each identified by masses of sellers posting the same price $p \in \mathbf{P}$, with $\mathbf{P} \subset \mathbb{R}_+$ being the set of prices. Sellers post DM prices before buyers enter the markets. All sellers commit to their posted prices. After observing all the posted prices, each buyer chooses the one that gives him the maximum surplus. Each seller can only produce for one buyer in each period. If a seller is visited by multiple buyers, he chooses one with equal probability. Let $n^* \leq N$ be the measure of active buyers in the DM. Let $n(p)$ be the market tightness in any submarket associated with p. In what follows we omit p as an argument in n. As before, the meeting rate for sellers is $\alpha(n)$, and $\alpha(n)/n$ for buyers in the submarket featuring p. By posting a lower price, a seller attracts more buyers and increases his trading probability. We seek a symmetric competitive search equilibrium in which all buyers and sellers are indifferent across submarkets. Without loss of generality, we can then focus on one submarket to solve for equilibrium as in Rocheteau and Wright (2005). In equilibrium the set of submarkets is complete so that no submarket could be created making some buyers and sellers better off.

The buyer's DM value function is now

$$V_t^b\left(p, n, a\right) = \frac{\alpha\left(n\right)}{n}\left(u - p\right) + W_t^b\left(a\right), \tag{16}$$

where p is the price posted by the chosen seller. From (3) and (16), buyers' value function is

$$W_t^b\left(a\right) = \Sigma + \left(\varphi_t + \rho\right) a + \beta W_{t+1}^b\left(0\right) + \max_{\hat{a},p,n}\left\{\beta\frac{\alpha\left(n\right)}{n}\left(u - p\right) + \left[\beta\left(\varphi_{t+1} + \rho\right) - \varphi_t\right]\hat{a}\right\}.$$

Since we solve for stationary equilibrium, we will omit the time subscript. Let $\bar{V}^b \equiv \max_{p^c \in \mathbf{P}}\{\frac{\alpha(n)}{n}\left(u - p^c\right) - sp^c\}$ be the equilibrium expected utility of a buyer in the DM.[16]

[15] For an extensive treatment of competitive search see the survey from Wright, Kircher, Julien and Guerrieri (2017).

[16] This is the *market utility* first used in McAfee (1993), Moen (1997), and Rocheteau and Wright (2005). It is the maximum expected utility buyers can get in any submarkets.

A seller takes $\bar{V}^b$ as given and solves

$$\tilde{V}^s(p,n) = \max_{p,n} \alpha(n)(p-c) \text{ s.t. } \tilde{V}^b(p,n) = \frac{\alpha(n)}{n}(u-p) - sp \geq \bar{V}^b,\ p \leq (\varphi+\rho)\,a. \quad (17)$$

The constraint $\tilde{V}^b(p,n) = \bar{V}^b$ determines the beliefs about market tightness n generated by p on an off-equilibrium path. This implies $dn/dp < 0$, i.e., lower price leads to higher participation on an off-equilibrium path.

Definition 2 *A stationary symmetric competitive search equilibrium, given s, is an allocation $\{p^c, n^*, \tilde{V}^s(p^c,n^*), \tilde{V}^b(p^c,n^*)\}$ such that:*

(i) sellers' optimality: $\tilde{V}^s(p^c,n) \geq \tilde{V}^s(p,n)$, s.t. $\tilde{V}^b(p,n) = \bar{V}^b$, $\forall p^c, p \in \mathbf{P}$;

(ii) buyers' optimality: $\tilde{V}^b(p,n) \leq \bar{V}^b$ and $n \geq 0$ with complementary slackness, and $\bar{V}^b \geq 0$, $\forall n > 0$;

(iii) beliefs are consistent with the measure of active buyers, i.e. $n = n^* \leq N$, in any submarket.

The first two points state that sellers and buyers are maximizing their expected payoffs given their beliefs, and active submarkets are associated with positive participation $n > 0$. The last point assures that beliefs are consistent with the aggregate measure of market tightness.

Given that prices are observed before buyers choose their asset holdings, we have $p = (\varphi+\rho)\,a$. Substituting p from the constraint yields

$$\max_n \alpha(n)\left[\frac{\alpha(n)\,u - n\bar{V}^b}{\alpha(n)+ns} - c\right]. \quad (18)$$

It is easy to show that the necessary condition of the above optimization problem is also sufficient. After using $\bar{V}^b$ from the necessary condition and the constraint, we derive the equilibrium price

$$p^c(s,n^*) = \frac{\alpha(n^*)\{[1-\varepsilon(n^*)]\,u + \varepsilon(n^*)\,c\} + \varepsilon(n^*)\,n^* sc}{\alpha(n^*) + \varepsilon(n^*)\,n^* s}, \quad (19)$$

where $\varepsilon(n) = \alpha'(n)n/\alpha(n)$ is the elasticity of the matching rate and $\varepsilon(n) < 1$.[17] In equilibrium, the market tightness is consistent with free entry

$$\frac{\alpha(n^*)}{n^*}(u - p^c) - sp^c = \bar{V}^b \geq 0. \tag{20}$$

Equations (19) and (20) generate $(p^c(s, n^*), n^*(s))$. Again, we study the existence and uniqueness of equilibrium by equating the aggregate demand and supply of liquidity, taking s as given, since it is an endogenous variable. Then, we back out asset price φ and participation n^* in equilibrium. The aggregate demand of liquidity $L^d(s) = n^*(s)p^c(s, n^*)$ is a function of the spread s. From $s = (r\varphi - \rho)/(\varphi + \rho)$, there is a bijection of asset price φ to s, we can solve for $\varphi = (1 + s)\rho/(r - s)$ with $r \neq s$, and the aggregate supply is $L^s(s) = (\varphi + \rho)A^s = (1 + r)\rho A^s/(r - s)$. The aggregate demand and supply of liquidity are characterized by the following two lemmas.

Lemma 3 *There exist $\bar{s}^c \geq r$ and $s^N \leq \bar{s}^c$, such that: (i) for $s \leq s^N$, $\exists!$ L^d with $n^* = N$, and $dL^d/ds < 0$; (ii) for generic $s \in (s^N, \bar{s}^c]$, $\exists!$ L^d with $n^* < N$, and $dL^d/ds < 0$; (iii) for $s > \bar{s}^c$, $\nexists$ $n^* > 0$ and L^d is not well-defined.*

Define ρ^N to be the dividend values corresponding to s^N . Let $\underline{\rho}$ be the cutoff for the existence of DM trade.

Recall s is the spread of assets and $\partial s/\partial\rho < 0$. As shown in Lemma 3, if the asset dividend is low enough, i.e., $\rho < \underline{\rho}$, and the cost of holding assets is high enough, the DM will shut down. As long as the DM operates and L^d is well-defined, it is monotonically decreasing in s. The DM participation of buyers varies depending on different values of ρ hence s. Now let ρ^F again denote the cutoff value of dividend under competitive search, above which assets are priced fundamentally. Next lemma characterizes the aggregate supply of liquidity.

[17] Interestingly, if we have divisible goods $q \in \mathbb{R}_+$ and sellers posting (p, q), we can obtain an additional equilibrium condition $u'(q)/c'(q) - 1 = s/\alpha(n)$, as the liquidity premium in Lagos and Wright (2005) with money. Thus, when asset is priced at its fundamental value, we get q^*, the efficient quantity. Furthermore, solving this equation for s and substituting it into (19) gives $p^c = \{[1 - \varepsilon(n^*)]\, u(q)c'(q) + \varepsilon(n^*)u'(q)c(q)\}/\{\varepsilon(n^*)\, u'(q) + [1 - \varepsilon(n^*)]c'(q)\}$, the standard pricing equation in a monetary environment. With bargaining, $\varepsilon(n^*)$ is replaced by the bargaining power parameter of buyers.

Lemma 4 *(i) For $\rho < 0$ $(r < s)$, L^s is convex and $dL^s/ds < 0$; (ii) for $\rho = 0$, L^s is perfectly elastic at $s = r$; (iii) for $\rho \in (0, \rho^F)$ $(0 < s < r)$, L^s is concave and $dL^s/ds > 0$; (iv) for $\rho \geq \rho^F$, L^s is perfectly elastic at $s = 0$.*

The Lemmas are illustrated in Figure 4, which shows the supply and demand for liquidity. As in the case for bargaining, we obtain multiplicity when $\rho < 0$.

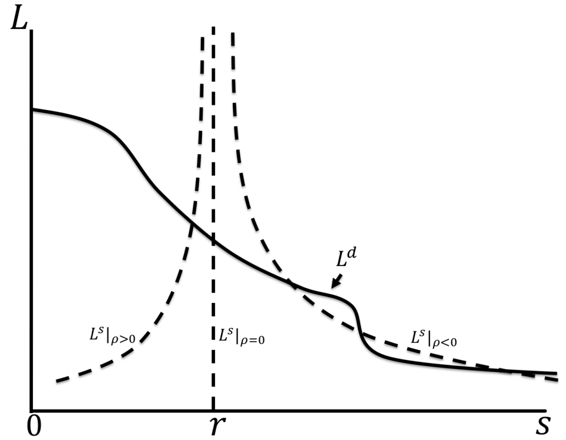

Figure 4: Supply and Demand for Liquidity under Competitive Search

Notice that the spread of assets can be rewritten in two parts, $s = r-(1+r)\rho/(\varphi+\rho)$. If $\rho = 0$ and assets have no dividend return, the second term vanishes and only the discount factor is left. Notice that now all the cutoff values of asset dividend are defined under competitive search.

Proposition 3 *In the model with competitive search, there exist $\rho^N < \rho^F$, and $\underline{\rho}$, such that: (i) for $\rho \geq \rho^F$, $\exists!$ symmetric SE with $\varphi = \varphi^F$ and $n^* = N$; (ii) for $\rho \in (\rho^N, \rho^F)$, $\exists!$ symmetric SE with $\varphi = \varphi^N > \varphi^F$ and $n^* = N$; (iii) for $\rho \in [\underline{\rho}, \rho^N]$, $\exists!$ symmetric SE if $\rho > 0$, and $\exists$ symmetric SE if $\rho < 0$, with $\varphi = \varphi^{n^*} > \varphi^F$ and $n^* \leq N$ ($<$ if $\rho < \rho^N$); (iv) for $\rho < \underline{\rho}$, $\nexists$ SE with an active DM.*

For $\rho > 0$, there is a unique equilibrium under both competitive search and bargaining, since there does not exist a trade-off between the probability to trade and the cost of holding liquidity. For $\rho < 0$, multiple equilibria exist with bargaining, but the equilibrium is still unique for $\rho \in (\rho^N, 0)$ with competitive search. This is because buyers search randomly and the equilibrium price in the bargaining game is the seller's reservation value, independent of the market tightness in the DM. With competitive search, the prices posted by sellers direct the buyers' search behavior and serve as a coordination device. We also obtain uniqueness if the cost of holding assets is a constant and independent of n, such as $s = r$ with $\rho = 0$. When holding assets is costless, i.e., $s = 0$, and the asset is priced at its fundamental value, equilibrium features the same price and participation in the DM as a credit economy.

Proposition 3 shows uniqueness for $0 < \rho < \rho^N$, i.e., $r > s > s^N$, while in Han et al. (2016), when money is the medium of exchange, there may still exist multiple equilibria for $r > i > i^N$. The different result is due to the cost of holding assets being endogenously determined while the cost of holding money is an exogenous policy variable. For $i > i^N$, the liquidity demand for money may not be unique for a countable number of interest rates. For these exogenous i, there are multiple equilibria featuring different real money balances. With assets, the liquidity demand can have multiple values at a countable number of s as well, but the spread is endogenously determined by $L^d = L^s$. According to Lemma 4, L^s is monotonically increasing in s, and there is a unique asset spread given n. For $\rho > 0$, the asset spread is increasing in n. With more buyers entering the DM, they face a higher cost of holding assets and a lower probability of trade. There is no coordination problem and a unique equilibrium n^* with a unique asset spread exists.

When $\rho < 0$ and $n^* < N$, we lose generic uniqueness, and this result is also different from the monetary economy. For a given spread s, one can show that equilibrium is generically unique, just as the monetary case, and sellers post only one price given the spread. However, because s is endogenously determined in our economy, there may exist equilibria with different spreads, each generating the same payoff for both buyers and sellers. For example, one equilibrium may feature a higher probability to trade and a higher spread of carrying assets, while another has a lower probability and a smaller cost of liquidity, making buyers indifferent in terms of equilibrium payoff and giving sellers the same profit. In this situation, competitive search still serves as a coordination device under the same spread, but there may exist equilibria with different values of s.

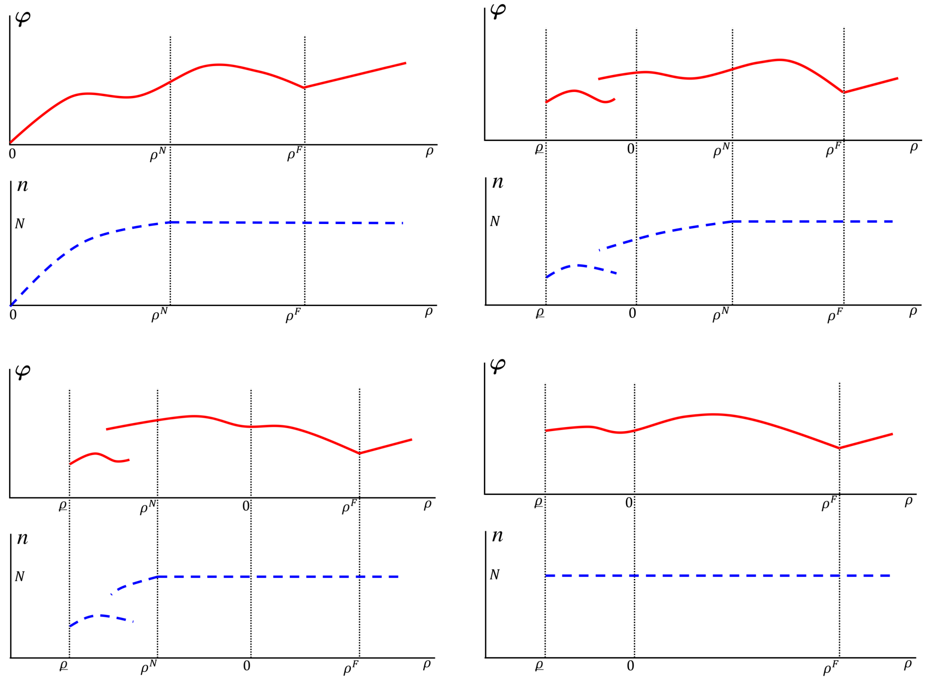

Figure 5: Competitive Search Equilibrium

Figure 5 shows the relationship between equilibrium participation n^* and dividend ρ by the dashed curves below the horizontal axis. Above the horizon, the solid curves represent asset price φ as a function of ρ. As long as the dividend is high enough, all buyers participate in the DM and assets are priced at the fundamental value. If ρ is smaller than ρ^N, not all buyers enter the DM. Since a larger dividend implies a smaller spread s, i.e., a lower cost of holding assets, the buyers' participation is monotonically increasing in ρ. However, the asset price φ may change in a non-monotonic way with respect to ρ. Equating the demand and supply of liquidity, we get the asset price $\varphi = L^d/A^s - \rho$, which is the difference between the dividend and the return of holding assets. As ρ gets larger, the asset return also increases due to a higher demand induced by ρ. Then, the change in asset price depends on how much the liquidity demand responds to ρ, which is ambiguous under general parameter values.

Let $\mathbf{A}$ again denote the amount of total liquidity in the market, and w.l.o.g. we fix asset supply A^s. The next proposition summarizes the effects of changing $\mathbf{A}$ under competitive search.

Proposition 4 *In the SE with competitive search: (i) for* $\mathbf{A} \geq \rho^F A^s$*, we have* $\partial\varphi/\partial\mathbf{A} > 0$ *and* $\partial n/\partial\mathbf{A} = 0$*; (ii) for* $\mathbf{A} \in [\rho^N A^s, \rho^F A^s)$*,* $\partial n/\partial\mathbf{A} = 0$ *and* $\partial\varphi/\partial\mathbf{A}$ *is ambiguous; (iii) for* $\mathbf{A} \in (\underline{\rho} A^s, \rho^N A^s)$*,* $\partial\varphi/\partial\mathbf{A}$ *and* $\partial n/\partial\mathbf{A}$ *are ambiguous.*

These findings are similar to the bargaining case. When liquidity is abundant in the economy, all buyers participate in the DM and assets are priced at the fundamental value. Hence, φ increases with $\mathbf{A}$. As the amount of liquidity decreases in the economy, the asset price exceeds its fundamental value due to liquidity premium, and φ may either increase or decrease with $\mathbf{A}$. Buyers begin to stop participating in the DM. For $\rho < 0$ and $n^* < N$, both the "hot potato" effect and the "poison apple" effect exist. Under general parameter values, either one may be the dominant force, and the effect of changing $\mathbf{A}$ on φ and n is ambiguous at different equilibrium.

3.3 Discussion

In the model used in this paper, the price of the good traded in a frictional market and the price of assets used in trade are inter-related. In this section we analyze the implications of modelling an indivisible DM good both for DM prices and for asset prices.

3.3.1 DM Prices

Here, we compare the results in an asset economy with a pure credit economy and a monetary economy, which are studied in Han et al. (2016). Indivisibility matters mainly from losing an intensive margin of adjustment. It makes the available surplus fixed without endogenous participation of buyers in the DM. In addition, different pricing mechanisms yield different results, and it matters if assets are used compared to pure credit or money.

To ease the comparison, we catalog the different cases. Let $n \leq N$ be the active measure of buyers in the DM. Let $B_l^j(n)$ be the buyers' benefit from participation, $l \in \{a, c, m\}$ be the three types of liquidity, assets, credit, or money, and $j \in \{b, c\}$ the type of pricing mechanisms, bargaining or competitive search. Let p^j be the equilibrium price under the mechanism j.

In the asset economy, we find

$$\begin{aligned} B^b_a(n) &= (u - p^b)\frac{\alpha(n)}{n} \geq sp^b, \\ B^c_a(n) &= (u - p^c)\frac{\alpha(n)}{n} \geq sp^c, \end{aligned}$$

where the spread $s(n, \rho)$ is decreasing in ρ and increasing (decreasing) in n if $\rho > 0$ (< 0). With bargaining, we find a unique n when $\rho > 0$, but when $\rho < 0$, there are two equilibrium n for a range of ρ. Similarly with competitive search, the asset equilibrium is unique for $\rho > 0$, and multiple equilibria exist for $\rho < 0$. The price reacts to the dividend value and endogenous participation.

In the credit economy, Han et al. (2016) finds that buyers participate in the DM if

$$\begin{aligned} B^b_c(n) &= \left(u - p^b\right)\frac{\alpha(n)}{n} \geq 0, \\ B^c_c(n) &= (u - p^c)\frac{\alpha(n)}{n} \geq 0. \end{aligned}$$

The main difference is the bargained price being independent of n, but not under competitive search. As long as $B^i_c(N) > 0$, all potential buyers participate in the DM.

For the monetary economy, Han et al. (2016) find

$$\begin{aligned} B^b_m(n) &= (u - p^b)\frac{\alpha(n)}{n} \geq ip^b, \\ B^c_m(n) &= (u - p^c)\frac{\alpha(n)}{n} \geq ip^c. \end{aligned}$$

Since $\alpha(n)/n$ is decreasing in n, under bargaining, for large enough i, $B^b_m(N) < ip^b$ and not all buyers participate in the DM. With competitive search, p^c is increasing in n and decreasing in i. Higher i reduces p^c, which increases $B^c_m(n)$, $\forall n$, but it also increases ip^c. This generates the potential for multiple equilibria with $n < N$. However, for generic values of i, these possibilities have zero measure. Thus, monetary equilibrium is generically unique.

3.3.2 Asset Prices

Indivisibility of the DM good matters for asset pricing. To discuss how, we compare our results with Lagos (2011) who uses divisible good in the DM, but like us, assets are traded in a frictionless and competitive market and used as a medium of exchange in the DM[18]. However, in his environment, money and assets are circulating in the goods market, while we focus on the use of asset only. He only considers positive value while we consider the possibility of negative dividend.[19]

In both papers, liquidity considerations affect asset prices as the asset price includes a liquidity premium when assets are scarce. More specifically, when the asset ameliorates trading frictions and is therefore valued as medium of exchange, the equilibrium asset price bears a liquidity premium (in excess of the fundamental value). In that case, only buyers are willing to hold the asset. When the economy does not have liquidity needs, the real asset is priced at its fundamental value and both buyers and sellers are willing to hold the asset as a store of value.

Again, indivisibility implies no intensive margin of adjustment in the good traded. However, we get an effect through the extensive margin. As the dividend value falls below ρ^N, the number of buyers entering the DM falls, until trade ceases at $\underline{\rho}$ when it is no longer profitable for buyers to acquire the asset and trade. Therefore, with fewer number of trades the economy is slowing down. It can be thought that this parameter region represents an economic recession. Dividend values are low, asset prices are decreasing and the overall level of trade is diminishing. Therefore, by modelling indivisible goods, we contribute to the study of the effect that asset prices can have on the exchange process and the overall economy.

The pricing mechanism matters for the study of assets and indivisible goods. With indivisible goods we get uniqueness of equilibria for all positive dividend values under both bargaining and price posting with directed search. Only in the case when the real return of the asset is negative does coordination failure emerge resulting in multiplicity for

[18] Other papers studying asset prices in the New Monetarist literature, such as Geramichalos et al (2007) and Lagos and Rocheteau (2009) assume that assets are traded in over-the-counter markets and focus on trading frictions in asset markets and its effect on asset prices.

[19] Lagos (2011) further assumes that the dividend value is determined stochastically.

both pricing mechanisms. Rabinovich (2017) studies equilibria when indivisible goods are traded using divisible assets as a medium of exchange under price posting with random search. He finds unique equilibria when assets are priced at its fundamental value. Away from fundamentals and with assets valued for their liquidity component, coordination failure emerges resulting in multiple equilibria.

Although we assume a constant dividend and focus on steady state, we can compare different trend periods in the link between asset prices and dividends. From Figures 2, 3 and 5, it is worth noting that the same asset price corresponds with multiple dividend and n values. The empirical literature has focused on the predictability of the dividend yield on asset prices. Nelson and Kim (1993) and Stambaugh (1999) argue that empirical analysis provide little evidence that the dividend yield can predict equity returns. Using the model we can analyze the effect of the dividend yield (i.e. ρ/φ) on asset prices and its expected return. In any equilibrium the asset price in (2) can be rewritten as

$$\frac{\varphi_{t+1}}{\varphi_t} = \frac{1}{\beta(1+s_t)} - \frac{\rho}{\varphi_t}.$$

When assets are priced fundamentally and when the economy is in recession, meaning that we are in the region between $\underline{\rho}$ and ρ^N, the model is unable to predict the effect of the dividend yield on asset prices. Therefore, consistent with empirical studies, we can show that for certain parameter values dividend yields are not a good predictor of asset prices. However, in the dividend region ρ^N to ρ^F, when assets are valued for its liquidity component, we find that a low dividend yield corresponds with high asset prices and high expected return. As Lewellen (2004) points out, generally a low dividend yield should predict that future returns are below average. However, in the U.S. in May 1995, the dividend yield reached a new low and studying equity prices, the NYSE index more than doubled over the period 1995-2001. Our model can support this empirical event. When assets are scarce and valued for its liquidity component then a low dividend yield translates into higher asset prices. This is for instance illustrated in Figure 2, by analyzing the region between ρ^N and ρ^F.

Finally, our model generates determinants of dividend yields. Imposing steady state on (2) and using (14) with bargaining, which holds when ρ is between ρ^N and ρ^F, we find

$$\frac{\rho}{\varphi} = \frac{Nc}{A^s} - 1.$$

For example, for fixed N and c, a decrease in A^s leads to higher dividend yield or lower price-dividend ratio. This is a movement along the asset price curve in region ρ^N to ρ^F in Figure 2. Note importantly that this conclusion is not robust to the trading mechanism used in the goods market. Figure 5 shows how under price posting and directed search, for the same region between ρ^N and ρ^F, the steady state asset price is non-monotonic in dividend value. This suggest that pricing mechanism used in consumption goods could contribute to the difficulties in predicting the link between dividend yield and asset price returns.

4 Lotteries

In an environment with indivisible goods, one can consider lotteries. To do so, let $E = \mathbf{P} \times \{0, 1\}$ denote the space of trading events, and $\mathbf{W}$ the Borel σ-algebra. Define a lottery to be a probability measure ω on the measurable space $(E, \mathbf{W})$. We can write $\omega(p, q) = \omega_q(q)\omega_{p|q}(p)$ where $\omega_q(q)$ is the marginal probability measure of q and $\omega_{p|q}(p)$ is the conditional probability measure of p on q. Without loss of generality, as shown in Berentsen et al. (2002), we restrict attention to $\tau = \Pr\{q = 1\}$ and $1 - \tau = \Pr\{q = 0\}$, and $\omega_{p|0}(p) = \omega_{p|1}(p) = 1$. Randomization is only useful on q because $\mathbf{Q}$ is non-convex. Thus, $\tau \in [0, 1]$ is the probability that the good is produced and traded.

When introducing lotteries, the generalized Nash bargaining problem becomes

$$\max_{p,\tau} (\tau u - p)^{\eta} (p - \tau c)^{1-\eta} \text{ s.t. } p \leq (\varphi + \rho)\, a,\ \tau \leq 1,$$

with $\tau u \geq p$ and $p \geq \tau c$.

Lemma 5 *The solution to the bargaining problem with lotteries is*

$$\left(p^b, \tau^b\right) = \begin{cases} \left(\bar{p}^b, 1\right) & \text{if } (\varphi+\rho)a > \bar{p}^b \\ ((\varphi+\rho)a, 1) & \text{if } \underline{p}^b \leq (\varphi+\rho)a \leq \bar{p}^b \\ \left((\varphi+\rho)a, (\varphi+\rho)a/\underline{p}^b\right) & \text{if } c \leq (\varphi+\rho)a < \underline{p}^b \\ (0,0) & \text{if } (\varphi+\rho)a < c \end{cases}$$

where $\bar{p}^b = (1-\eta)\,u + \eta c$ *and* $\underline{p}^b = uc/(\eta u + (1-\eta)c)$*.*

The buyer's CM value is

$$W_t^b(a) = \Sigma + (\varphi_t + \rho)a + \beta W_{t+1}^b(0) + \beta \max_{\hat{a}} v(\hat{a}),$$

where $v(\hat{a}) = (\tau^b u - p^b)\alpha(n)/n - s(\varphi_{t+1} + \rho)\hat{a}$.

Proposition 5 *In the model with bargaining and lotteries: (i) for* $\rho \geq \rho^F$ *and* $\rho > 0$, $\exists!$ *SE with* $\varphi = \varphi^F$ *and* $n^* = N$*; (ii) for* $\rho \in [\rho^N, \rho^F)$ *and* $\rho > 0$, $\exists!$ *SE with* $\varphi = \varphi^N > \varphi^F$ *and* $n^* = N$*; (iii) for* $\rho \in (\underline{\rho}, \rho^N)$ *and* $\rho > 0$*, or* $\rho = \underline{\rho} < 0$, $\exists!$ *SE with* $\varphi = \varphi^{n^*} > \varphi^F$ *and* $n^* < N$*; (iv) for* $\rho \in (\underline{\rho}, 0)$, $\exists$ *SE; (v) for* $\rho < \underline{\rho}$, $\nexists$ *SE with an active DM; (vi)* $p^b = \underline{p}^b$ *and* $\tau^b = 1$ *hold for (i)-(iv).*

Lotteries are not used in equilibrium, and buyers bring enough assets to achieve the maximum expected surplus from trade at $\tau^b = 1$. In stationary equilibrium, $\underline{p}^b$ and τ^b do not change with ρ. The buyer's asset holding is always just enough to pay for the DM transaction, which is not affected by the spread s. Compared to the case of bargaining without lotteries, we still get a continuum of equilibria for $\rho \in (\underline{\rho}, 0)$, since the coordination problem still exists. Lotteries do not lead to uniqueness of equilibrium.

Finally, introducing lotteries to competitive search, the price posting problem becomes

$$\begin{aligned} \tilde{V}^s(p,n,\tau) &= \max_{p,\tau,n} \alpha(n)(p - \tau c) \\ \text{s.t. } \tilde{V}^b(p,n,\tau) &= \frac{\alpha(n)}{n}(\tau u - p) - sp \geq \bar{V}^b,\ p \leq (\varphi+\rho)\,a,\ \tau \leq 1. \end{aligned}$$

The following proposition shows that lotteries are not used in competitive search equilibrium either.

Proposition 6 *In the model with competitive search and lotteries, there exist ρ^F, ρ^N, and $\underline{\rho}$, such that: (i) for $\rho \geq \rho^F$, $\exists!$ symmetric SE with $\varphi = \varphi^F$ and $n^* = N$; (ii) for $\rho \in (\rho^N, \rho^F)$, $\exists!$ symmetric SE with $\varphi = \varphi^N > \varphi^F$ and $n^* = N$; (iii) for $\rho \in [\underline{\rho}, \rho^N]$, $\exists!$ symmetric SE if $\rho > 0$, and $\exists$ symmetric SE if $\rho \leq 0$, with $\varphi = \varphi^{n^*} > \varphi^F$ and $n^* \leq N$ ($<$ if $\rho < \rho^N$); (iv) for $\rho < \underline{\rho}$, $\nexists$ SE with an active DM; (v) $\tau^c = 1$ holds for (i)-(iii).*

5 Conclusion

In this paper, we use a general equilibrium model to study asset pricing based on consumption decision of indivisible goods in frictional markets. Indivisibility matters, especially when terms of trade in the goods market are determined by bargaining. The bargained price gives sellers no surplus and is independent of the asset dividend. Introducing lotteries does not change the independence on dividend, but sellers are able to extract a positive surplus. Under competitive search, the trading price depends on the asset dividend and the number of buyers in the market. Lotteries do not matter under competitive search.

Under bargaining, the equilibrium asset price is unique as long as the asset dividend is non-negative. With a negative dividend, we find two equilibria, with low and high participation. The congestion nature of the matching technology in the good's market generates a concave net benefit in the number of active buyers, leading to a coordination problem and two equilibria. With price posting and competitive search, we find a unique asset price equilibrium for positive dividends, and multiple equilibria exist for negative dividends. While in a monetary economy, using price posting as a coordination device can solve the problem present under bargaining, it cannot eliminate multiplicity when assets are the medium of exchange. This is because the cost of holding assets is endogenously determined, but the cost of holding money is exogenous.

Overall, the consequences of indivisibility on the goods side matter and differ from indivisibility on the asset side. Indivisibility affects the bargaining outcome because it isolates the price of goods from dividend value and the number of buyers. Price posting with competitive search reestablishes the link and generically produces a unique equilibrium with certain dividend values. While we have focused on stationary equilibrium, the model can easily be used to study asset price dynamics. We leave this for future research.

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Appendix

Proof of Lemma 1. Given buyers' participation constraint, $n^* = N$ if $\frac{\alpha(n^*)}{n^*}(u-c) - sc > 0$. Define $s^N = \frac{\alpha(N)}{N}\frac{u-c}{c}$, then given $s \leq s^N$, for all $n^* < N$, $\frac{\alpha(n^*)}{n^*}(u-c) - sc > 0$, contradiction. Hence $n^* = N$, and hence (i). Define $\bar{s}^B = \frac{u-c}{c}$, then $\forall n^* \in (0, N]$, for $s > \bar{s}^B$, $\nexists\, n^*$ st $\frac{\alpha(n^*)}{n^*}(u-c) - sc \geq 0$, hence (iii). For $s \in (s^N, \bar{s}^B]$, $\frac{\alpha(n^*)}{n^*}(u-c) - sc = 0$. Then $dn^*/ds = \frac{\partial[\alpha(n^*)/n^*]}{\partial n^*}\frac{c}{u-c} < 0$, therefore $dL^d/ds < 0$, and hence (ii). ■

Proof of Lemma 2. If assets are priced at the fundamental value, then all buyers participate in the DM and $s = 0$. Let $\rho^F = (1-\beta)c/A$. If $\rho \geq \rho^F$, then $\forall\, n$ st $(\varphi + \rho)A^s/n \geq (\varphi^F + \rho)A^s/n \geq \rho^F A/(1-\beta) = c$. The liquidity need for assets is satisfied and the marginal holders of assets only care about the store of value function of assets. Hence, $\varphi = \varphi^F$ and $s = 0$, hence (iv). If $\rho = 0$, the cost of holding assets is $s = r$, hence (ii). Otherwise, $\varphi = (1+s)\rho/(r-s)$, then substitute s into the liquidity supply and $L^s = (1+r)\rho A^s/(r-s)$, with $\partial L^s/\partial s = (1+r)\rho A^s/(r-s)^2$ and $\partial^2 L^s/\partial s^2 = -2(1+r)\rho A^s/(r-s)^3$. It is easy to check $\partial L^s/\partial s > 0$ and $\partial^2 L^s/\partial s^2 < 0$ for $\rho \in (0, \rho^F)$, i.e., $0 < s < r$, hence (iii) and for $\rho < 0$, $\partial L^s/\partial s < 0$ and $\partial^2 L^s/\partial s^2 > 0$. Hence (i). ■

Proof of Proposition 1. Figure 1 illustrates the liquidity demand and the liquidity supply with bargaining. There may exist a unique or multiple intersections of demand and supply, and we need to discuss different cases. First, case (i) is straightforward from Lemma (2). In this case, L^s and L^d don't have an intersection for all $s > 0$. $L^s \geq L^d$. Therefore sellers hold some assets too. We have $s = 0$, $\varphi = \varphi^F$, and $n^* = N$. The equilibrium is unique. For all $\rho < \rho^F$, all equilibria satisfy $L^s = L^d$ and the buyers' participation constraint. Rewrite the constraint we get $-\rho \leq \frac{n}{A^s}\left[\frac{\alpha(n)}{n}\beta(u-c) - (1-\beta)c\right] = \frac{n}{A^s}B(n) \equiv f(n)$. Notice $f''(n) < 0$, then $f(n)$ has a unique global maximum point on the support $[0, N]$. Now define $\underline{\rho} = -\max_{n\in[0,N]} f(n)$. For $\rho < \underline{\rho} \leq 0$, $f(n) < -\rho\ \forall n > 0$, then the buyers' participation constraint doesn't hold and the DM shuts down, $L^s = L^d$ will never hold, hence (vii). For (ii), (iii), (iiii), (v), and (vi); we need to examine the uniqueness and the number of active buyers.

We establish the uniqueness first. For $\rho = 0$, the asset case is equivalent to the fiat money case with zero money growth rate, and we show the uniqueness in Han et al. (2016) proposition 3. For $\rho > 0$, we have $dL^s/ds > 0$ and $dL^d/ds \leq 0$, hence the equilibrium is unique. For $\rho = \underline{\rho}$, the equilibrium is unique because of the unique n which maximizes

$f(n)$. For $\rho \in (\underline{\rho}, 0)$, there are two roots st $f(n) = -\rho$, call them n_l^* and n_h^*. With the lose of generality, let $n_l^* < n_h^*$. Then it is easy to show $n_l^* < N$, then $n^* = n_l^*$ which satisfies $f(n_l^*) = -\rho$ and $L^s = L^d$ is an equilibrium. We focus on the other root n_h^*. If $n_h^* \geq N$, we have $f(N) \geq f(n_h^*) = -\rho$. Then $n^* = N$ and $L^s = L^d$ is the other equilibrium; otherwise, $n^* = n_h^* < N$ and $L^s = L^d$ is the other equilibrium. In sum, for $\rho \in (\underline{\rho}, 0)$, the two equilibria with $n^* = n_l^*$ and $n^* = \max\{n_h^*, N\}$, and it is easy to show the equilibrium with a higher n^* has a higher φ. Then we examine the number of active buyers. Define $\rho^N = -\frac{N}{A^s}\left[\frac{\alpha(N)}{N}\beta(u-c) - (1-\beta)c\right]$, then *for* $\rho \in [\rho^N, \rho^F)$, $f(n) > -\rho$ $\forall\ n < N$, hence $n^* = N$ is a possible candidate equilibrium. If $\rho < \rho^N$, all equilibria should satisfy $n^* < N$. After considering uniqueness and the number of buyers, we have (ii), (iii), (iiii), (v), and (vi). ■

Proof of Proposition 2. W.l.o.g., we take A^s as given. For the unique equilibrium or the equilibrium with higher participation, we have shown that there exists a cutoff ρ^F such that $\rho^F A^s = (1-\beta)cN$. Then, $\forall \mathbf{A} > (1-\beta)cN$, we have $\varphi = \varphi^F$ and $n = N$, and then $\partial\varphi/\partial\mathbf{A} > 0$ and $\partial n/\partial\mathbf{A} = 0$, hence (i). For $\rho \in [\rho^N, \rho^F)$, i.e., $\mathbf{A} \in [\rho^N A^s, \rho^F A^s)$, we have $\mathbf{A} \geq (1-\beta)cN - \beta(u-c)\alpha(N)$, and then $n = N$ and $\partial\varphi/\partial\rho < 0$, implying $\partial n/\partial\mathbf{A} = 0$ and $\partial\varphi/\partial\mathbf{A} < 0$, hence (ii). For $\rho \in [\underline{\rho}, \rho^N)$, i.e., $\mathbf{A} \in [\underline{\rho}A^s, \rho^N A^s)$, $\partial\varphi/\partial\rho > 0$ and $\partial n/\partial A^s > 0$, and thus $\partial n/\partial\mathbf{A} > 0$ and $\partial\varphi/\partial\mathbf{A} > 0$, hence (iii). For the equilibrium with lower participation and $\rho \in (\underline{\rho}, 0)$, i.e., $\mathbf{A} \in (\underline{\rho}A^s, 0)$ we have $\partial\varphi/\partial\rho < 0$ and $\partial n/\partial A^s < 0$, and hence $\partial n/\partial\mathbf{A} < 0$ and $\partial\varphi/\partial\mathbf{A} < 0$. ■

Proof of Lemma 3. To prove that L^d is a well-defined function for $s \leq \bar{s}^C$, it is sufficient to show $n^* > 0$ exists and is unique. Substituting p^c into (20) gives $\alpha\varepsilon(u-c)s + \alpha^2\varepsilon(u-c)/n^* = \alpha[(1-\varepsilon)u + \varepsilon c]s + \varepsilon n^* cs^2$. Define $h(n^*, s) = \alpha\varepsilon(u-c)s + \alpha^2\varepsilon(u-c)/n^* - \alpha[(1-\varepsilon)u + \varepsilon c]s - \varepsilon n^* cs^2$. Given any $n \in (0, N]$, $h(n, s) = 0$ is a quadratic function in s, which has two real solutions with opposite signs. The positive solution s_+, satisfying $h(n, s_+) = 0$, is an implicit function of n, $s_+(n)$. Let $s_+(0) = \lim_{n\to 0} s_+(n) < \infty$, and $s_+(0)$ is continuous on $[0, N]$. Define s^N by $h(N, s^N) = 0$ and $\bar{s}^C = \max_{n\in[0,N]} s_+(n)$. For $s < s^N$, $h(N, s) > 0$ hence $n^* = N$. Then $L^d = Np^c(N, s)$ is unique, and $dL^d/ds = Ndp^c(N, s)/ds < 0$, hence (i). For $s > \bar{s}^c$, $h(n^*, s) < 0$ $\forall n^*$, and the free-entry condition does not hold due to $\alpha(n^*)(u - p^c)/n^* - sp^c < 0$, hence (iii).

Regarding (ii), for $s \leq \bar{s}^c$, $h(n^*, s) = 0$ always holds for some $n^* > 0$, and L^d exists.

To show that L^d is generically unique and monotone, consider $L^d = n^* p^c$ and $dL^d/ds = \partial L^d/\partial s + (\partial L^d/\partial n^*)(\partial n^*/\partial s)$. Given $h(n^*, s) = 0$, we have $L^d = \alpha(n^*)n^*u/[\alpha(n^*) + sn^*]$, hence $\partial L^d/\partial s < 0$ and $\partial L^d/\partial n^* > 0$. Then, it is sufficient to show that n^* is generically unique and $\partial n^*/\partial s < 0$. We claim that although there might be multiple n^* which maximize $\tilde{V}^s(n, s)$, n^* is still unique and $\partial n^*/\partial s < 0$ for generic s. To see this, suppose $\tilde{V}^s(n_1^*, s) = \tilde{V}^s(n_2^*, s) = \max_n \tilde{V}^s(n, s)$ and $n_2^* > n_1^*$. Then, n_1^* is the minimum n maximizing $\tilde{V}^s(n, s)$, and $\tilde{V}^s(n_1^*, s) > \tilde{V}^s(n, s)$, $\forall n < n_1^*$. For $\epsilon > 0$ small enough, $\tilde{V}^s(n_1^*, s+\epsilon) > \tilde{V}^s(n, s+\epsilon)$ also holds for $n < n_1^*$ due to continuity. If $\partial^2\pi/\partial s\partial n^* < 0$, then $\tilde{V}^s(n_1^*, s+\epsilon) > \tilde{V}^s(n_2^*, s+\epsilon)$, and the global maximizer is a unique n in the neighborhood of n_1^*. Next, we need to show $\partial^2\tilde{V}^s/\partial s\partial n^* < 0$. Derive $\partial\tilde{V}^s/\partial n$ from (18),

$$\frac{\partial \tilde{V}^s}{\partial n} = \frac{(\alpha + sn)\left[(u-c)\,\alpha' - sc\right] - s\,(1-\varepsilon)\left[(u-c)\,\alpha - snc\right]}{(\alpha + sn)^2/\alpha}.$$

Define $T(s) = (\alpha+sn)[(u-c)\alpha' - sc] - s(1-\varepsilon)[(u-c)\alpha - snc]$, and $T'(s) = n[(u-c)\alpha' - sc] - (\alpha+sn)c - (1-\varepsilon)[(u-c)\alpha - snc] + snc(1-\varepsilon)$. Since $T_{n=n^*} = 0$, $\partial^2\pi/\partial s\partial n^* = T'(s)/[(\alpha+sn^*)^2/\alpha]$. With $\alpha(u-c) - sn^*c > 0$, we have

$$T'_{n=n^*}(s) = \frac{-\left[\alpha\,(u-c) - sn^*c\right](1-\varepsilon)\,\alpha - c\,(\alpha+sn^*)\,(\alpha+sn^*\varepsilon)}{\alpha + sn^*} < 0.$$

Therefore, $\partial^2\tilde{V}^s/\partial s\partial n^* < 0$ holds. In addition, $\arg\max_n \tilde{V}^s(n, s)$ might have more than one solution for some $s \geq s^{NC}$, but the set of such asset spreads has measure zero, hence (ii). Finally, we prove $\bar{s}^c \geq r$ by contradiction. Suppose $\bar{s}^c < r$, then for $s_1 = (r\varphi_1 - \rho_1)/(\varphi_1 + \rho_1) \in (\bar{s}^c, r)$, $\rho_1 > 0$ and $n_1^* = 0$. Hence, $\varphi_1 = \varphi_1^F$ and $s_1 = 0$, contradicting $s_1 > \bar{s}^c > 0$. ∎

Proof of Lemma 4. If assets are priced at the fundamental value, then all buyers participate in the DM and $s = 0$. Let $\rho^F = (1-\beta)p^c_{N,s=0}/A$. If $\rho \geq \rho^F$, the average asset holding $(\varphi + \rho)A^s/n \geq (\varphi^F + \rho)A^s/n \geq \rho^F A/(1-\beta) = p^c_{N,s=0}$. The liquidity need for assets is satisfied and the marginal holders of assets only care about the store of value function. Hence, $\varphi = \varphi^F$ and $s = 0$. If $\rho = 0$, the cost of holding assets is $s = r$. If $\rho < \rho^F$ and $\rho \neq 0$, substitute s into the liquidity supply and $L^s = (1+r)\rho A^s/(r-s)$, with $\partial L^s/\partial s = (1+r)\rho A^s/(r-s)^2$ and $\partial^2 L^s/\partial s^2 = -2(1+r)\rho A^s/(r-s)^3$. It is easy to check $\partial L^s/\partial s > 0$ and $\partial^2 L^s/\partial s^2 < 0$ for $\rho \in (0, \rho^F)$, and for $\rho < 0$, $\partial L^s/\partial s < 0$ and

$\partial^2 L^s/\partial s^2 > 0$. ■

Proof of Proposition 3. Figure 4 illustrates the liquidity demand and the liquidity supply with competitive search. There may exist a unique or multiple intersections of demand and supply, and we need to discuss different cases. For $\rho \geq \rho^F$, a downward-sloping L^d and a perfectly elastic L^s ensure the existence and uniqueness of equilibrium s^* with $n^* = N$, hence (i). For $\rho = 0$, assets are equivalent to money with zero inflation, and the proof follows Proposition 5 in Han et al. (2016). For $\rho \in (0, \rho^F)$, L^d and L^s intersect once and there exists a unique equilibrium. For $\rho < 0$, $\bar{s}^C \geq r$ according to Lemma 3. If $\bar{s}^C = r$, $\nexists$ non-degenerate equilibrium; if $\bar{s}^C > r$, L^d and L^s may have more than one intersection, hence more than one candidate equilibrium. Given n^* being a function of s, we can rewrite the seller's problem (18) as

$$\max_s \alpha\left(n^*\left(s\right)\right)\left[\frac{\alpha\left(n^*\left(s\right)\right)u - n^*\left(s\right)\bar{V}^b}{\alpha\left(n^*\left(s\right)\right) + n^*\left(s\right)s} - c\right].$$

Given different values of s^* satisfying the first-order condition, there could be more than one s^* which maximize seller's profit. Hence $\nexists$ uniqueness in this region. Next is to show the existence of $\underline{\rho}$. If $\bar{s}^c = r$, $\underline{\rho} = 0$. Consider $\bar{s}^c > r$. $s \leq r$ implies $\rho \geq 0$, and thus (iii). For $s \in (r, \bar{s}^c)$, $\rho < 0$, $\partial L^s/\partial \rho = (1+r)A^s/(r-s) < 0$, and L^d is constant. Hence, $\exists!$ $\rho^*(n)$ such that $L^s(\rho^*) = L^d$, and define $\underline{\rho} = \min_{s\in[r,\bar{s}^C]} \rho^*(s) < 0$. For $\rho < \underline{\rho}$, $L^s(\rho) > L^d$, and there exists no equilibrium, hence (iv).

For the rest of the proposition on participation and asset prices, first consider $\rho \geq \rho^F$. According to Lemma 4, the cost of holding assets $s = 0$, implying $\varphi = \varphi^F$ and $n^* = N$. Let $\rho^N = (r - s_N)p^c(s^N, N)/(1+r)A$. If $\rho \in (\rho^N, \rho^F)$, then $s^N > s > 0$. The buyer's participation constraint is slack, and $(\varphi + \rho)A^s/N = p^c(s^N, N)$. Hence, $n^* = N$ and $\varphi = \varphi^N = (1+s)p^c(s^N, N)/(1+r)A > \varphi^F$. If $\rho \in [\underline{\rho}, \rho^N]$, the buyer's participation constraint is binding, and $s > 0$ and $(\varphi + \rho)A^s/n^* = p^c$. Therefore, $\varphi = \varphi^{n^*} = n^*(1+s)p^c/N(1+r)A > \varphi^F$. ■

Proof of Proposition 4. Similar to bargaining, A^s is taken as given. There exists a cutoff ρ^F satisfying $\rho^F A^s = (1-\beta)\, N[1-\varepsilon(N)]u + (1-\beta)\, N\varepsilon(N)c$. Then, $\forall \mathbf{A} > \rho^F A^s$, we have $\varphi = \varphi^F$ and $n = N$, and then $\partial\varphi/\partial\mathbf{A} > 0$ and $\partial n/\partial\mathbf{A} = 0$, hence (i). For $\rho \in (\rho^N, \rho^F)$, i.e., $\mathbf{A} \in (\rho^N A^s, \rho^F A^s)$, we have $\mathbf{A} \geq (1-\beta)\, Np^c(s^N, N) - \beta[u - p^c(s^N, N)]\alpha\,(N)$, then $n = N$ and $\partial\varphi/\partial\rho$ is ambiguous, implying $\partial n/\partial\mathbf{A} = 0$ and $\partial\varphi/\partial\mathbf{A}$ is ambiguous, hence (ii). For $\rho \in [\underline{\rho}, \rho^N]$, i.e., $\mathbf{A} \in [\underline{\rho} A^s, \rho^N A^s]$, $\partial\varphi/\partial\rho$ and $\partial n/\partial\rho$ are ambiguous, and thus $\partial\varphi/\partial\mathbf{A}$ and $\partial n/\partial\mathbf{A}$ are ambiguous, hence (iii). ■

Proof of Lemma 5. Using λ_1 and λ_2 for the multipliers on the asset constraint and the lotteries constraint gives the following Kuhn-Tucker conditions.

$$
\begin{aligned}
0 &= -\eta\,(\tau u - p)^{\eta-1}\,(p - \tau c)^{1-\eta} + (1-\eta)\,(\tau u - p)^{\eta}\,(p - \tau c)^{-\eta} - \lambda_1 && (21)\\
0 &= \eta u\,(\tau u - p)^{\eta-1}\,(p - \tau c)^{1-\eta} - c\,(1-\eta)\,(\tau u - p)^{\eta}\,(p - \tau c)^{-\eta} - \lambda_2 && (22)\\
0 &= \lambda_1\,((\varphi + \rho)a - p)\\
0 &= \lambda_2\,(1 - \tau)\,.
\end{aligned}
$$

It is straightforward to check that if $\lambda_1 = 0$, $p = \tau^B \bar{p}^B$. Substituting this into (22) implies $\lambda_2 > 0$, and hence $\tau^B = 1$. In order to support $\tau^B = 1$, buyer needs to bring enough asset to the DM trade, i.e. $(\varphi + \rho)a > \bar{p}^B$. On the other hand, if $\lambda_2 = 0$, $\tau^B = p^B/\underline{p}^B$. Substituting this into (21) implies $\lambda_1 > 0$ and $p^B = (\varphi + \rho)a$. In order to satisfy $\tau^B < 1$, we need $(\varphi + \rho)a < \underline{p}^B$. If both λ_1 and λ_2 are greater than zero, $p^B = (\varphi + \rho)a$ and $\tau^B = 1$. $\lambda_1 > 0$ implies $(\varphi + \rho)a < \bar{p}^B$, and $\lambda_2 > 0$ implies $(\varphi + \rho)a > \underline{p}^B$. Finally, the seller certainly does not trade if he meets a buyer with $(\varphi + \rho)a < c$. ■

Proof of Proposition 5. First, buyers do not want to bring $(\varphi_{t+1} + \rho)\hat{a} > \bar{p}^b$, since additional assets do not affect the surplus from trade. Second, they do not bring $(\varphi_{t+1} + \rho)\hat{a} < c$, for no trade. Next, for $(\varphi_{t+1} + \rho)\hat{a} \in (\underline{p}^b, \bar{p}^b)$, $v'(\hat{a}) = -(\varphi_{t+1} + \rho)[s + \alpha(n)/n] < 0$, and buyers want to choose $(\varphi_{t+1} + \rho)\hat{a} = \underline{p}^b$. For $(\varphi_{t+1} + \rho)\hat{a} \in (c, \underline{p}^b)$, $v'(\hat{a}) = (\varphi_{t+1} + \rho)\,[\alpha(n)\eta(u - c)/nc - s]$, and the sign of $v'(\hat{a})$ depends on the value of the spread s. Since $\alpha(n)(u - \underline{p}^b)/n - s\underline{p}^b = \underline{p}^b[\alpha(n)\eta(u - c)/nc - s]$, $v'(\hat{a})$ shares the same sign as $\alpha(n)(u - \underline{p}^b)/n - s\underline{p}^b$. Suppose $v'(\hat{a}) < 0$, buyers choose $\tau^b = 0$ and there is no equilibrium with an open DM. If $v'(\hat{a}) > 0$, buyers of measure n in the DM choose $(\varphi_{t+1} + \rho)\hat{a} = \underline{p}^b$. The cutoff spread satisfying $v'(\hat{a}) = 0$ is given by $\alpha(n)(u - \underline{p}^b)/n - s\underline{p}^b = 0$, which is equivalent to the participation constraint $n[\alpha(n)\beta(u - \underline{p}^b)/n - (1-\beta)\underline{p}^b]/A^s = g(n) \geq -\rho$. Since $g''(n) < 0$, let $\underline{\rho} = -\max_n g(n)$, $\rho^F = (1-\beta)\underline{p}^b/A$, and $\rho^N = [(1-\beta)\underline{p}^b - \beta\alpha(N)(u - \underline{p}^b)/N]/A$. For $\rho \geq \underline{\rho}$, all equilibria feature $p^b = \underline{p}^b$ and $\tau^b = 1$. If $\rho \geq \rho^F$, then $\varphi = \varphi^F$; otherwise $\varphi > \varphi^F$. If $\rho \geq \rho^N$, then $n^* = N$; otherwise $n^* < N$. The rest of the proof on equilibrium stability follows directly from Proposition 1. ■

Proof of Proposition 6. We need to check that sellers always post $\tau^c = 1$ and the rest of the proof follows Proposition 3. Let λ be the multiplier for τ, and the FOCs are

$$\begin{aligned} 0 &= \varepsilon(n)(p - \tau c) - \frac{\alpha(n)[1 - \varepsilon(n)](\tau u - p)}{\alpha(n) + ns}, && (23) \\ 0 &= \tau\left[\frac{\alpha^2(n)u}{\alpha(n) + ns} - \alpha(n)c - \lambda\right], && (24) \\ 0 &= \lambda(1 - \tau). \end{aligned}$$

Given the buyer's optimal participation $n = n^*$ and (23), we have

$$p^c = \frac{\alpha(n^*)\{[1 - \varepsilon(n^*)]\tau u + \varepsilon(n^*)\tau c\} + \varepsilon(n^*)n^* s\tau c}{\alpha(n^*) + \varepsilon(n^*)n^* s}.$$

Solve for λ from (24), and we need $\lambda = \alpha(n^*)(u - c) - cn^*s > 0$ to assure $\tau^c = 1$. Since $p^c/\tau > c\ \forall\tau$, $\alpha(n^*)(u - c) - cn^*s > \alpha(n^*)(u - p^c/\tau) - n^*sp^c/\tau \geq 0$. The last inequality is the buyer's participation constraint in the DM, which holds if $\rho \geq \underline{\rho}$ and $n^* > 0$. ■