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Intermediation in Over-the-Counter Markets with Price Transparency

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Intermediation in Over-the-Counter Markets with Price Transparency*

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Abstract

A salient feature of over-the-counter (OTC) markets is intermediation: dealers buy from and sell to customers as well as other dealers. Traditionally, the search-theoretic literature of OTC markets has rationalized this as a consequence of random meetings and ex post bargaining between investors. We show that neither of these are necessary conditions for intermediation. We build a model of a fully decentralized OTC market in which search is directed and sellers post prices ex ante. Intermediation arises naturally as an equilibrium outcome for a broad class of matching functions commonly used in the literature. We also explore the sorting implications of the equilibrium pattern of trade and show that it is constrained efficient.

JEL Classification: G11, G12, G21, D83

Keywords: Over-the-counter markets, search frictions, intermediation, price transparency, competitive search, sorting

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1 Introduction

A large number of securities, including government, municipal, and corporate bonds; asset-backed securities; federal funds; and various types of derivatives, are traded in decentralized or "over-the-counter" (OTC) markets. A prevalent feature of OTC markets is *intermediation*: trades are intermediated by dealers who buy from and sell to customers, as well as other dealers.[1] Traditionally, the theoretical literature building off of the influential work of Duffie et al. (2005, 2007) has rationalized extensive intermediation as a mechanical consequence of the OTC markets' decentralized nature (finding a counterparty to trade takes time) and opaqueness (the trading parties negotiate the terms of trade privately). In recent years, however, OTC markets have become far more transparent due to several regulatory initiatives, as well as the growing use of electronic trading platforms allowing traders at once to obtain quotes at which many dealers are willing to trade.[2] Yet, intermediation remains a key feature of OTC markets, as shown by the recent empirical evidence in Adrian et al. (2017) and Bessembinder et al. (2020). This presents a challenge for the traditional mechanism behind intermediation found in the theoretical literature.[3]

To address this challenge, we develop a fully decentralized search-theoretic model of an OTC market with price transparency. In the model, heterogeneous asset owners post

[1] The literature has proposed several definitions of intermediation. In this paper, we follow Hugonnier et al. (2014, 2018) and say an agent acts as an intermediary if she is actively trying to buy the asset when she does not have it and actively trying to sell it when she does. Empirical evidence for intermedation in OTC markets abounds; see, among others, Duffie (2011) for an overview, Bessembinder and Maxwell (2008) for the corporate bonds market, Afonso and Lagos (2014, 2015) for the federal funds market, and Li and Schürhoff (2019) for the municipal bonds market.

[2] These initiatives, which require require the timely public dissemination of post-trade price and volume information, include the Municipal Securities Rulemaking Board (MSRB) for the municipal bonds market, the Trade Reporting and Compliance Engine (TRACE) for the corporate bonds and securitized assets markets, as well as the recent MiFID II set of regulations mimicking TRACE for the European corporate bonds markets. For evidence on electronic platforms, see Stafford (2016), Liu et al. (2018) and Vogel (2019).

[3] A few recent papers highlight the importance of including additional heterogeneity in investors' characteristics for explaining the empirically observed "core-periphery" structure of many OTC markets. In all of these papers, intermediation is a consequence of random meetings and ex post bargaining. See the literature review section for a detailed comparison with our approach.

publicly available prices at which they are willing to sell the asset and heterogeneous non-owners choose to which price to direct their orders. Both asset prices and contact rates between investors are endogenous and reflect investors' heterogeneous liquidity needs. We show that intermediation arises naturally in this framework where agents *choose* their trading counterparties based on available prices.

The modeling environment follows the literature initiated by Duffie et al. (2005). The economy is populated by infinitely-lived investors who are either owners or non-owners of a single indivisible asset in fixed supply. Investors are heterogeneous in their valuation of the asset; we assume there are finitely many valuations and refer to each as an investor's type. In addition, investors experience periodic preference shocks that change their type and this creates incentives to trade. As in Duffie et al. (2005) and many others, we restrict agents inventory holdings to either 0 or 1 units of the asset.

There are two major points of departure from Duffie et al. (2005): i) the asset market is purely decentralized and ii) there is price transparency. The first departure allows us to study the trading pattern of the asset market as an endogenous equilibrium outcome. In this regard, we follow the approach of Hugonnier et al. (2014) and postulate that all trades in the economy take place in bilateral meetings without the aid of a perfectly competitive dealer market.[4] In particular, the roles of agents as either customers (those who only buy or sell the asset) or dealers (those who intermediate trades between other agents) are endogenous in our model. The second departure, namely price transparency, allows us to study whether the economy would feature intermediation when prices are publicly available. To incorporate price transparency, we model search frictions using the competitive search protocol. Specifically, we follow Lester et al. (2015) and allow owners to post prices at which they are willing to sell the asset. Non-owners observe all posted prices and choose to direct

[4] Other papers that feature a purely decentralized OTC market with a similar modeling approach include Afonso et al. (2014), Hugonnier et al. (2018), Üslü (2019) and Bethune et al. (2019).

their search towards a specific price.

The competitive search equilibrium of our fully decentralized OTC market has several novel features. To begin with, agents endogenously segment into different submarkets. We refer to the collection of all asset owners who post a particular price and all non-owners who direct their orders towards that same price as a submarket. Moreover, we refer to owners and non-owners who participate in a submarket as call sellers and buyers, respectively. Even though buyers observe prices, there are still frictions which preclude the immediate execution of trades, due to various institutional, informational, and technological constraints.[5] We capture these frictions through the means of a matching function — the expected execution time is an exponentially distributed random variable with mean governed by the queue length on the market (the ratio of buyers to sellers).[6] Since buyers observe all prices, they direct their orders only to counterparties with whom they have gains from trade. Consequently, all meetings result in trades. Thus, each submarket is characterized by a price and a contact rate, both of which are endogenous objects. This is in sharp contrast to most of the existing literature, which has modeled OTC markets through the means of random bilateral meetings with exogenous contact rates.

In equilibrium, investors fully segment into different submarkets. That is, each submarket contains only one type of seller and one type of buyer. This is because when sellers post prices, they face a trade-off between posting high prices which attract orders at a low arrival rate, or posting low prices which attract orders at a high arrival rate. This is a classic trade off found in competitive search models; see Wright et al. (2019) for a comprehensive review. We show that in equilibrium no two buyer-seller pairs of different type trade in the same

[5] In practice, there are several reasons that contribute to the delay of order execution between traders. Some of these include: informational asymmetries that may lead to collateral verification; institutional constraints that may put limits on the volume and number of trades; and even technological constraints, such as the speed with which computers and electronic platforms show and process order information.

[6] This modeling approach is standard in the labor search literature. For papers in the OTC literature that use the matching function, see Lagos and Rocheteau (2007, 2009) and Lester et al. (2015).

submarket. Furthermore, we show that the equilibrium prices segment the market efficiently. That is, the submarkets that open and the order execution speeds therein coincide with the planner's allocation. In this regard, we generalize the constrained efficiency result of Lester et al. (2015) in a fully decentralized setting.

Prices also play an important role for how parties split the surplus of trade. When a low valuation owner transfers the asset to a high valuation non-owner, this creates a net gain in the combined utility flow of the two investors. This net gain is what we refer to as the *fundamental surplus*. The trade surplus, however, has an additional component which is due to investors' capacity to act as *intermediaries* by selling the asset when they have it and buying the asset when they do not have it, conditional on keeping the same asset valuation. When a non-owner receives the asset, she can in turn post it for sale herself, which results in potential gains from transferring the asset to a non-owner of even higher valuation. At the same time, the owner who just sold the asset can direct a buy order on a different submarket. While she waits for her order to be executed, she foregoes the utility flow from owning the asset. This foregone utility together with the purchase price constitute the owner's "cost" of restocking her inventory. Both the non-owner's option of reselling and the owner's cost of restocking affect the surplus of the trade: if the non-owner can sell the asset quickly at a high price, this increases the surplus; if the owner can restock her inventory quickly at a low price, this also increases the surplus. As a result, both the gains and the costs of intermediation are shared between the two parties through the price.

The main result of this paper is that in our model, intermediation arises naturally, even though agents can *choose* their trading counter-parties. Intuitively, low type owners have, in principle, a stronger incentive to trade with higher valuation non-owners because this trade creates large fundamental surplus. However, if the terms of trade are favorable enough, mid-valuation non-owners can entice low valuation owners to trade with them, instead of with high valuation non-owners. In particular, if the price is high enough, then the low valuation owners can extract all of the gains that a mid-valuation non-owner can receive from reselling the asset to the high valuation non-owner. In this situation, the only cost of selling to a mid-type non-owner vis-à-vis a high type one is an opportunity cost: waiting for an order from a mid-valuation non-owner precludes from waiting for one from a high valuation non-owner. Then, if the order arrival rate for mid-type non-owners is high enough, the owner will always have an incentive to trade with them, since she trades quickly and extracts all of the surplus — including all potential gains of trade the mid-valuation non-owner realizes from reselling the asset to a high valuation non-owner. Symmetrically, high valuation non-owners have, in principle, stronger incentive to buy from low valuation owners. However, if the terms of trade are favourable enough, they may buy from mid-valuation owners instead. In particular, if the price is set at the cost of restocking the owner's inventory, the high type non-owner extracts all of the surplus when trading with a mid-type owner. Then, if the order execution speed is fast enough, she will trade with the mid-valuation owner instead of the low valuation one. Whether or not such behavior arises in equilibrium depends on the functional form of the matching function. We show that for a broad class of matching functions which satisfy Inada-type conditions on the order arrival and execution rates, all investors — except for those with the lowest and highest valuation of the asset — intermediate trades in equilibrium.

Our final contribution is to connect the search-theoretic framework for the study of OTC markets with the sorting literature, recently reviewed in Chade et al. (2017). Specifically, since our model features two-sided heterogeneity and agents who choose their counterparties, characterizing the equilibrium is a sorting problem. We are able to derive predictions regarding whether agents with high asset valuations trade with agents with high (negative assignment) or low valuations (positive assignment).[7] To the best of our knowledge, we are

[7] In the sorting literature, an assignment is referred to as positive (negative) if high valuation buyers match with high (low) valuation sellers. In this context, the ordering of buyer and seller types is such that the match surplus is increasing in both types. On the contrary, in our model the fundamental surplus is decreasing in the seller's valuation. As a result, for our purposes, what the literature has traditionally referred to as a "high type seller" corresponds to an owner with low valuation of the asset.

the first to derive sorting implications in an OTC framework. In contrast with the results of Shi (2001) and Eeckhout and Kircher (2010a), we find that the equilibrium assignment in our model is very often not *strictly* monotone, in the sense that a trader of a certain valuation may match with traders of multiple valuations. There are two reasons our results differ from the existing literature. First, whereas Shi (2001) assumes free entry of agents in one side of the market, we follow the OTC literature and assume fixed masses of agents. Second, whereas Eeckhout and Kircher (2010a) assume that agents' valuations are drawn from a twice continuously differentiable CDF, we use discrete distributions. Thus, in our economy, when the relative mass of sellers (buyers) of a certain type is large enough they can meet the asset demand (supply) of multiple types of buyers (sellers). This often results in an imperfect assignment, which is ruled out in most papers in the sorting literature.[8]

Related Literature. This paper contributes to the fast growing search-theoretic literature on OTC markets initiated by Duffie et al. (2005, 2007); for recent surveys, see Lagos et al. (2017) and Weill (2020). Most closely related to ours are the papers which feature a fully decentralized asset market. Hugonnier et al. (2014) provide a general analysis of a fully decentralized benchmark model with $\{0, 1\}$ asset holdings, an arbitrary distribution of investor valuations, exogenous random meetings, and bilateral bargaining. Neklyudov (2019) adds heterogeneity in contact rates to explain the observed heterogeneity in bid-ask spreads across different dealers. Geromichalos and Herrenbrueck (2016a,b) merge a decentralized OTC market with the Lagos and Wright (2005) framework. Afonso and Lagos (2015) and Üslü (2019), building upon Lagos and Rocheteau (2009), analyze the case with unrestricted asset holdings. In addition to unrestricted asset holdings, Üslü (2019) allows for rich heterogeneity in investors' preferences, inventories, and contact rates. He finds that heterogeneity in contact rates is the main driver of intermediation patterns. The major difference between these pioneering contributions and our work is that intermediation in these papers is a me-

[8] See Jerez (2014) for an exception.

chanical consequence of random meetings and price opaqueness, whereas in our framework intermediation is the result of price transparency and investors' optimal choices of trading counterparties.

A growing body of work aims to explain intermediation patterns by introducing additional heterogeneity in the search-theoretic OTC framework with price opaqueness.[9] Farboodi et al. (2017a) consider heterogeneity in bargaining power, whereas Farboodi et al. (2017b) consider heterogeneity in contact rates. In both papers, the structure proposed by Duffie et al. (2005) emerges endogenously, with the agents with superior bargaining power or higher contact rates acting as intermediaries. In Bethune et al. (2019), investors are heterogeneous in their ability to learn the private valuation of the asset held by others (screening ability). This results in heterogeneous informational rents and, consequently, agents who possess superior screening ability emerge as intermediaries. In contrast to all these papers, ours features price transparency and directed search, maintaining traditional differences in investor valuations as the only source of heterogeneity. Our results imply that neither random search nor price opaqueness, nor additional heterogeneity are necessary conditions for intermediation to arise in this class of models.

Developing an alternative approach, Chang and Zhang (2018) build a bilateral matching model that combines elements of the search-theoretic literature with network models of OTC markets. In their model, agents choose whom to meet with, but cannot observe other agents' asset valuation prior to forming a match. As a result, not all meetings lead to trade, and prices are determined only after matches are formed. They show that certain agents endogenously emerge as intermediaries and form the core of the trading network. Our paper shares a common insight with their work: when search is directed intermediation emerges endogenously. However, we derive this result in the context of the workhorse search and

[9] To be precise, these papers aim to explain the "core-periphery" structure found in many real-world OTC markets, a research question outside the scope of this paper.

matching environment initiated by Duffie et al. (2005). As a result, we show that one does not have to resort to alternative ways of modeling frictions to obtain intermediation when agents choose their trading counterparties.

Finally, our work is part of the voluminous literature on competitive search which has followed the seminal contributions of Moen (1997), Acemoglu and Shimer (1999), and Mortensen and Wright (2002). Within that strand of literature, the most closely related papers to ours are the ones which feature two-sided heterogeneity such as Shi (2001), Shimer (2005), Eeckhout and Kircher (2010a), and Jerez (2014). Competitive search has been widely used in many applications, comprehensively surveyed in Wright et al. (2019). However, there are relatively few applications of competitive search in the context of OTC markets. As mentioned above, Lester et al. (2015) provide a detailed analysis in the context of semi-centralized model featuring a frictionless market for dealers. Since we consider a fully decentralized OTC market, our paper extends their analysis of competitive search in OTC markets to include the issues of intermediation and sorting. Armenter and Lester (2017) employ competitive search in a two-period model of the federal funds market which has heterogeneity only in one side of the market. Lastly, a series of papers use the tractability of competitive search to analyze issues of asymmetric information in asset markets. These include Guerrieri et al. (2010), Guerrieri and Shimer (2014, 2018), Chang (2017), Li (2019), and Williams (2019).

2 The Model

Agents, assets, and preferences. Time is continuous and runs forever. The economy is populated by a unit measure of infinitely-lived, risk-neutral investors who discount the future at rate $r > 0$. There is one durable asset in fixed supply $A \in (0, 1)$ and one perishable numéraire good with marginal utility normalized to one. Investors can hold either zero or one unit of the asset, which is assumed to be indivisible. We refer to investors who have the asset as "owners" and investors who do not as "non-owners". The instantaneous utility flow investors receives from holding the asset is δ_i, where $i \in \{1, 2, ..., I\} \equiv \mathcal{I}$ indexes an investor's type with $1 < I < \infty$, and $\delta_i > \delta_j$ for $i > j$. Types change over time: each investor receives i.i.d preference shocks according to a Poisson process with intensity γ. Conditional on receiving a preference shock, the investor draws a new type j from some discrete cumulative distribution function $F(\delta_j)$ with support $\mathcal{I}$. We denote the probability mass function of that distribution by $f(\delta_j)$. The measures of owners and non-owners type i are $o(\delta_i)$ and $n(\delta_i)$ respectively.

Matching and trade. Investors interact in a purely decentralized market: all trades takes place in bilateral meetings. The market features complete price transparency: each owner posts (and commits to) a publicly available price at which she is willing to sell the asset to a non-owner. Non-owners observe all available prices and direct their orders to at most one of the available prices. In this regard we follow Moen (1997), Acemoglu and Shimer (1999), and Eeckhout and Kircher (2010a) and model the market using a *competitive search* protocol. Moreover, we assume the types of all agents are publicly observable. We refer to the collection of all owners posting the same price and all non-owners willing to buy the asset at this price as a "submarket".[10] Owners (non-owners) who participate in some submarket we call sellers (buyers) on that submarket. Owners (non-owners) who do not participate in any submarket, we refer to as idle.

In practice, it takes time to execute trades between investors because of frictions. For example, there might be informational asymmetries (it takes time to verify collateral), institutional constraints (on the volume of trade), and technological limitations on the speed of execution.[11] We follow the previous literature on OTC markets and capture these delays

[10] Moen (1997) and Mortensen and Wright (2002) assume that third-party market makers set up submarkets and promise a price and expected waiting time for transactions to any agents that show up. We follow the formalization of Acemoglu and Shimer (1999) and Eeckhout and Kircher (2010a) in which sellers post terms of trade and buyers self-select to different submarkets. These are all equivalent interpretations of the competitive search protocol; see Wright et al. (2019).

[11] For example, Hugonnier et al. (2018) point out that most trades on the inter-dealer municipal bond

through search frictions, which prevent instantaneous trading between investors. The speed with which counter-parties trade is endogenous in our model and depends on the queue length (the ratio of buyers and sellers) at each submarket.[12] In particular, we follow Lagos and Rocheteau (2009) and Lester et al. (2015) assume that the number of trades is given by a matching function.[13]

Formally, suppose there is a measure $o(\delta_i, p)$ of owners type i posting a particular price p and a measure $n(\delta_j, p)$ of non-owners type j interested in acquiring a unit of the asset at this price. Then, at each instant, the flow of trades executed is given by $m\left(\sum_i o(\delta_i, p), \sum_j n(\delta_j, p)\right)$, a function which has constant returns-to-scale, is strictly increasing, strictly concave and twice continuously differentiable with respect to its two arguments. As is standard, the waiting time for an owner to sell her unit of the asset is an exponentially distributed random variable with parameter $\lambda(q(p)) \equiv \frac{m\left(\sum_i o(\delta_i,p), \sum_j n(\delta_j,p)\right)}{\sum_i o(\delta_i,p)} = m\left(1, q(p)\right)$, where $q(p) \equiv \frac{\sum_j n(\delta_j,p)}{\sum_i o(\delta_i,p)}$ is the queue length on the submarket. Symmetrically, the waiting time for a non-owner is exponentially distributed with intensity $\frac{\lambda(q(p))}{q(p)}$. The assumptions on the matching function imply that $\lambda(\cdot)$ is continuous, strictly increasing, and strictly concave.

market are still being executed over the phone.

[12] To be precise, the queue length equals the ratio of non-owners over owners along the equilibrium path. Out of the equilibrium path, the queue length will be determined by a condition on investors' beliefs; we explain the condition precisely later in this Section 3.1.

[13] This modeling technology has been used extensively in search models of the labor market; it is usually referred to as the "matching function". Here, we use it as a modeling device that uses the measures of investors as inputs and gives the final flow of trades as output.

3 Equilibrium

3.1 Value Functions and Equilibrium Definition

Owners. Let $V_1(\delta_i, p)$ denote the expected lifetime payoff of an owner of type i who posts a price p, expecting a queue length $q(p)$:

$$rV_1(\delta_i, p) = \delta_i + \gamma \sum_j \left[V_1^*(\delta_j) - V_1(\delta_i, p)\right] f(\delta_j) + \lambda(q(p)) \left[p - V_1(\delta_i, p) + V_0^*(\delta_i)\right]. \quad (1)$$

Intuitively, the owner enjoys a utility flow δ_i from holding the asset until she i) receives a preference shock or ii) sells the asset. At a rate γ, the owner draws a new preference type j with probability $f(\delta_j)$. In this event, she obtains the maximum attainable utility of being an owner of type j, denoted by $V_1^*(\delta_j)$, and loses her current expected payoff, $V_1(\delta_i, p)$. At a rate $\lambda(q(p))$, she meets a non-owner who purchases the asset at price p. In this event, the owner receives the price and the maximum attainable utility of being a non-owner of type i, denoted by $V_0^*(\delta_i)$, but loses her current expected payoff, $V_1(\delta_i, p)$.

Non-owners. We can similarly characterize the value function for a non-owner with type i who participates in submarket p. A non-owner who finds all posted prices too high can choose to not participate in any market, i.e. she can choose the price of $p = \emptyset$. This allows us to keep notation concise and denote the value function of non-owners who are active buyers in some submarket and non-owners who are idle by $V_0(\delta_i, p)$. In particular,

$$rV_0(\delta_i, p) = \gamma \sum_j \left[V_0^*(\delta_j) - V_0(\delta_i, p)\right] f(\delta_j) + \frac{\lambda(q(p))}{q(p)} \left[V_1^*(\delta_i) - p - V_0(\delta_i, p)\right]. \quad (2)$$

Intuitively, a non-owner with type i does not enjoy a positive utility flow since she does not own the asset. Two distinct events can affect her value function: i) the preference shock and ii) meeting an owner. At rate γ, the non-owner draws a new preference type and this type is

j with probability $f(\delta_j)$. In this event, she obtains the maximum attainable utility of being a non-owner of type j, denoted by $V_0^*(\delta_j)$, but loses her current expected payoff, $V_0(\delta_i, p)$. At rate $\lambda(q(p))/q(p)$, she meets an owner and purchases the asset. In that case, she receives the maximum attainable utility of being an owner of type δ_i, denoted by $V_1^*(\delta_i)$, but loses her current expected payoff, $V_0(\delta_i, p)$. If the non-owner chooses the option of $p = \emptyset$, her matching rate with an owner is 0, i.e. $\lambda(q(\emptyset))/q(\emptyset) = 0$.

The maximum utilities an investor can attain are formally defined as:

$$V_1^*(\delta_i) = \sup_{p \in \mathcal{P}} \left\{ V_1(\delta_i, p) \right\}, \tag{3}$$

$$V_0^*(\delta_i) = \sup_{p \in \mathcal{P}} \left\{ V_0(\delta_i, p) \right\}, \tag{4}$$

for owners and non-owners correspondingly, where $\mathcal{P}$ is the set of prices posted in equilibrium.

Out-of-equilibrium beliefs. So far, the value functions are only determined along the equilibrium path, since the queue length, $q(p)$, is well defined only for prices $p \in \mathcal{P}$. We follow Eeckhout and Kircher (2010a) and Jerez (2014) by imposing restrictions on beliefs in the spirit of subgame perfection: we assume that owners expect a positive queue length when posting a given price only if there is some non-owner who is willing to buy the asset at that price. Moreover, owners expect the largest possible queue length for which they can find such a non-owner type, i.e. they expect non-owners to queue in the submarket until it is no longer profitable to do so. Formally, the queue length satisfies the following condition for any price:

$$q(p) = \sup \left\{ q' \in \mathbb{R}_+ : \exists \delta; \quad V_0(\delta, p) \geq \max_{p' \in \mathcal{P}} V_0(\delta, p') \right\}, \tag{5}$$

or $q(p) = 0$, if the set is empty. This belief restriction defines queue lengths and value functions on the entire domain of prices, not only along the equilibrium path.

Laws of motion. The masses of owners and non-owners evolve due to preference shocks and asset trading. In particular,

$$\dot{o}(\delta_i) = \gamma f(\delta_i) \sum_{j \neq i} o(\delta_j) - \gamma o(\delta_i) \sum_{j \neq i} f(\delta_j) - \sum_{p \in \mathcal{P}} o(\delta_i, p) \lambda(q(p)) + \sum_{p \in \mathcal{P}} n(\delta_i, p) \frac{\lambda(q(p))}{q(p)}, \quad (6)$$

$$\dot{n}(\delta_i) = \gamma f(\delta_i) \sum_{j \neq i} n(\delta_j) - \gamma n(\delta_i) \sum_{j \neq i} f(\delta_j) + \sum_{p \in \mathcal{P}} o(\delta_i, p) \lambda(q(p)) - \sum_{p \in \mathcal{P}} n(\delta_i, p) \frac{\lambda(q(p))}{q(p)}. \quad (7)$$

At rate γ, owners of any given type are hit with a preference shock. Conditional on that event, each agent draws type i with probability $f(\delta_i)$. Hence, the first term represents the net flow of agents into $o(\delta_i)$ due to preference shocks. The second term captures the net flow out due to preference shocks: each owner of type i receives a preference shock at rate γ and, conditional on that event, her type changes with probability $\sum_{j \neq i} f(\delta_j)$. In each submarket p there are $o(\delta_i, p)$ owners of type i selling the asset. Each of them executes a trade at rate $\lambda(q(p))$. Once the trade is executed the owner transfers the asset to the non-owner and so she becomes a non-owner of type i. Thus, the third term in equation (6) captures the flow of agents out of $o(\delta_i)$ due to trade. Analogously, the last term captures the flow into owners of type i due to non-owners of type i purchasing the asset. The law of motion for non-owners is given by (7) and the intuition behind it is analogous.

Equilibrium definition. A steady state equilibrium is a set of value functions $V_1^*(\delta_i)$ and $V_0^*(\delta_i)$, with $i \in \mathcal{I}$, a set of prices $\mathcal{P}$, a set of masses $o(\delta_i, p)$ and $n(\delta_i, p)$, with $p \in \mathcal{P}$ and $i \in \mathcal{I}$, and a queue length function $q(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, such that owners choose prices that maximize their values defined by (1), non-owners choose submarkets that maximize their values defined by (2), the queue length function satisfies condition (5), equations (6) and (7) hold with $\dot{o}(\delta_i) = \dot{n}(\delta_i) = 0$, for all $i \in \mathcal{I}$, the accounting identities $\sum_{p \in \mathcal{P}} o(\delta_i, p) = o(\delta_i)$ and $\sum_{p \in \mathcal{P}} n(\delta_i, p) = n(\delta_i)$ hold for all $i \in \mathcal{I}$ and the resource constraints $\sum_{i \in \mathcal{I}} o(\delta_i) = A$ and $\sum_{i \in \mathcal{I}} n(\delta_i) = 1 - A$ hold.

Value functions along the equilibrium path. The equilibrium payoff of each investor is determined by her own prices and submarket choices and by the *distribution* of price and submarket choices in the economy, which in turn has to be consistent with the optimal choices of individual investors. In equilibrium, an individual owner of type i takes the distributions of investor masses $o(\delta_i, p)$ and $n(\delta_i, p)$ as given and, according to the equilibrium definition, her pricing decision solves $\max_p V_1(\delta_i, p)$ (taking into account that her choice of price affects the expected queue length, q). The owner can post a price that attracts a zero ($q(p) = 0$) or positive ($q(p) > 0$) measure of non-owners. The response of non-owners to this seller's pricing decision is captured by (5), which holds by assumption outside the set of equilibrium prices $\mathcal{P}$ and, by the equilibrium definition, inside $\mathcal{P}$. Thus, the problem of an owner is

$$\max_{q,p} \{\lambda(q)\,[p - V_1(\delta_i, p) + V_0^*(\delta_i)] : q = \sup\{q' \in \mathbb{R}_+ : \exists\delta;\ V_0(\delta, p) \geq V_0^*(\delta)\}\}\,. \tag{8}$$

It is easy to establish that, by continuity of $V_0(\delta_i, p)$ in (8), any optimally set price leaves the non-owners who queue up in this submarket their market utility $V_0^*(\delta)$. As a result, the owner's i problem reduces to

$$\max_{q,p,j} \{\lambda(q)\,[p - V_1(\delta_i, p) + V_0^*(\delta_i)] : V_0(\delta_j, p) = V_0^*(\delta_j)\}\,. \tag{9}$$

Variants of equation (9) are at the core of most competitive search models. Intuitively, in equilibrium, owners optimally choose prices and non-owner type(s) they wish to attract, such that non-owners receive exactly the utility they would get if they were to participate in a different submarket.[14] $V_0^*(\delta)$ is an endogenous object but is taken as given by individual investors. The existing literature has used this strategy of characterizing the equilibrium (often referred to as *market utility approach*) extensively; see Wright et al. (2019) for a recent review.

[14] Notice that a submarket featuring a price and a queue length that do not satisfy the constraint in (9) would attract zero non-owners, since it does not provide non-owners with their outside option.

The Bellman equations along the equilibrium path read:

$$(r+\gamma)V_1^*(\delta_i) = \delta_i + \gamma V_1^* + \lambda(q^*(\delta_i))\left[p^*(\delta_i) - \Delta V^*(\delta_i)\right], \tag{10}$$

$$(r+\gamma)V_0^*(\delta_i) = \gamma V_0^* + \frac{\lambda(q^{**}(\delta_i))}{q^{**}(\delta_i)}\left[\Delta V^*(\delta_i) - p^{**}(\delta_i)\right], \tag{11}$$

where $V_1^* = \sum_j V_1^*(\delta_j)f(\delta_j)$ and $V_0^* = \sum_j V_0^*(\delta_j)f(\delta_j)$ are the average maximum utilities across investor types, $p^*(\delta_i)$ is an optimally chosen price posted by owner i that attracts queue length $q^*(\delta_i)$ in equilibrium, $p^{**}(\delta_i)$ and $q^{**}(\delta_i)$ are the equilibrium price and queue length in a submarket that non-owner i has optimally chosen to participate and $\Delta V^*(\delta_i) = V_1^*(\delta_i) - V_0^*(\delta_i)$ is the *reservation value* of investor i (that is, the expected utility of owing minus the expected utility of not owing the asset) along the equilibrium path.[15]

Reservation values along the equilibrium path. We can use the expressions for the Bellman equations, (10) and (11), to express the reservation value of investor δ_i as

$$(r+\gamma)\Delta V^*(\delta_i) = \delta_i + \gamma\Delta V^* + \lambda(q^*(\delta_i))\left[p^*(\delta_i) - \Delta V^*(\delta_i)\right] - \frac{\lambda(q^{**}(\delta_i))}{q^{**}(\delta_i)}\left[\Delta V^*(\delta_i) - p^{**}(\delta_i)\right]. \tag{12}$$

This expression highlights the dual role of investors (as buyers and sellers) in the market. The first two terms capture the utility flow of owning the asset and the expected capital gain (loss) from switching types. The last two terms capture the search options of the investor when she is an owner and a non-owner. When she does not have the asset, the investor can queue on the submarket $p^{**}(\delta_i)$ and, if matched, she purchases the asset in exchange for the price. She then receives her reservation value and is now an owner who can exercise her option to post the asset for sale. If the optimally chosen $p^*(\delta_i)$ attracts a positive queue length, once the investor makes contact with a buyer, she receives the price but loses her reservation value. Thus, she is now a non-owner and can again exercise her search option of

[15] Of course, there may be more than one prices and more than one submarkets that maximize an owner's and a non-owner's lifetime utility respectively.

looking to buy the asset.

Following Hugonnier et al. (2014), we call this behavior *intermediation*. It is important to stress that intermediation in our model is a choice. The sellers on submarket $p^{**}(\delta_i)$ choose to post terms of trade such that they trade with an investor type i who then sells the asset to buyers on submarket $p^*(\delta_i)$, even though sellers on market $p^{**}(\delta_i)$ can post terms of trade that attract the non-owners on market $p^*(\delta_i)$. Similarly, non-owners on market $p^*(\delta_i)$ choose to buy the asset from an investor type i rather than to queue directly on the market $p^{**}(\delta_i)$. Thus, any intermediation that takes place in our economy is an outcome of investors' optimal choices regarding which type to direct their orders towards. In contrast, intermediation in the benchmark theoretical model which features random search, Hugonnier et al. (2014), is a mechanical consequence of the matching technology.[16]

Trade surplus along the equilibrium path. Whenever investors trade, this generates a surplus. We denote it by $S^*(\delta_i, \delta_j)$, where i, j are the types of the seller and buyer respectively and the star superscript represents the value along the equilibrium path. If these two agents trade, the buyer gains her reservation value, but transfers the price to the seller who in turn loses her reservation value. Hence, $S^*(\delta_i, \delta_j) = \Delta V^*(\delta_j) - \Delta V^*(\delta_i)$. Using (12) the surplus can be rewritten as

$$\begin{aligned} S^*(\delta_i, \delta_j) = \frac{\delta_j - \delta_i}{r+\gamma} + \frac{\lambda(q^*(\delta_j))}{r+\gamma}\left[p^*(\delta_j) - \Delta V^*(\delta_j)\right] + \frac{\lambda(q^{**}(\delta_i))}{q^{**}(\delta_i)(r+\gamma)}\left[\Delta V^*(\delta_i) - p^{**}(\delta_i)\right] \\ - \frac{\lambda(q^*(\delta_i))}{r+\gamma}\left[p^*(\delta_i) - \Delta V^*(\delta_i)\right] - \frac{\lambda(q^*(\delta_i))}{q^*(\delta_i)(r+\gamma)}\left[\Delta V^*(\delta_j) - p^*(\delta_i)\right]. \quad (13) \end{aligned}$$

The above expression highlights how the option of acting as an intermediary affects the

[16] In Hugonnier et al. (2014) investors with moderate valuations intermediate trades for investors with extreme valuations, but this is an implication of random meetings taking place with an exogenous intensity. Moderate types meet relatively frequently with both non-owners with higher types than their own and owners with lower types than their own and, as a result, intermediate a large fraction of trades. In our model, low types *choose* to search for and sell to moderate types and moderate types *choose* to search for and sell to high types. Hence, intermediation is not a by-product of random exogenous meetings and asset misallocation across investor types, but rather arises naturally as investors direct their search towards different prices.

trade surplus. The first term of (13) captures the discounted difference in the utility flows of the two investors. We call this the *fundamental surplus*. The next two terms capture the potential benefit for the two investors if they do act as intermediaries. When the buyer receives the asset she not only gains her flow value but also the option to post the asset for sale on a different submarket $p^*(\delta_j)$. At the same time the owner i gains the option to search for the asset her self and reacquire it at the price $p^{**}(\delta_i)$. Both of these options have positive payoffs and so they increase the surplus of the match. If the buyer can sell the asset at favourable terms quickly or if the seller can reacquire the asset quickly at a low price, this increases the surplus of their match. The last two terms capture the loss of the investors' outside option once they trade. Once the asset is transferred the owner can no longer post it for sale and the buyer loses the option to search for the asset.

3.2 Equilibrium Characterization

Next, we proceed to characterize the equilibrium (we defer equilibrium existence to Section 3.5). The first set of equilibrium properties establishes that investors' value functions and reservation values are increasing in investor types, a result which is standard in the literature.

Lemma 1. *For any $i > j$, the following inequalities hold: $\Delta V^*(\delta_i) > \Delta V^*(\delta_j)$; $V_0^*(\delta_i) \geq V_0^*(\delta_j)$; $V_1^*(\delta_i) > V_1^*(\delta_j)$. Furthermore, if some non-owners of type j are active in some submarket, then $V_0^*(\delta_i) > V_0^*(\delta_j)$.*

Next, we can use equations (10) and (11), along with the constraint $V_0(\delta, p) = V_0^*(\delta)$, to express the owner's problem (9) more compactly:

$$\max_{q,j} \left\{ \lambda(q) S(\delta_i, \delta_j) - q \left[(r+\gamma) V_0^*(\delta_j) - \gamma V_0^* \right] \right\}, \tag{14}$$

where $S(\delta_i, \delta_j) = \Delta V^*(\delta_j) - \Delta V(\delta_i, p)$. The first order condition with respect to the queue length results in an expression for the price of trade between i and j, familiar from the competitive search literature:

$$p(\delta_i, \delta_j) = \eta(q)\Delta V^*(\delta_j) + [1 - \eta(q)]\Delta V(\delta_i, p), \tag{15}$$

where $\eta(q) \equiv \frac{om_o}{m}$ is the elasticity of the matching function with respect to the total measure of owners in this submarket. In equilibrium, the price of the asset is a weighted average of the two equilibrium reservation values and thus captures investors' option to act as intermediaries. The reservation value of the buyer is higher if she can sell the asset quickly at a favorable price, once she has acquired it. This increases the surplus of the match and consequently the price. As a result some of the potential benefits the buyer receives by acting as an intermediary are passed to the seller at the time of the trade. Similarly, if the seller has to reacquire the asset, once she has sold it, at relatively high prices or slow speeds, the expected "costs" or restocking her inventory are high. This decreases the match surplus and the price. Thus, some of the expected costs of reacquiring the asset are passed to the buyer.

It is instructive to compare (15) with equation (4) of Hugonnier et al. (2014): they are exactly the same, except the weights on $\Delta V^*(\delta)$ and $\Delta V(\delta_i, p)$ are the bargaining powers of owners and non-owners respectively instead of the matching function elasticity.[17] In the model of Hugonnier et al. (2014), the price of trade is the result of Nash bargaining between owners and non-owners with different bargaining powers. In both models, the price is a weighted sum of the reservation values of investors who trade, but the weights are different: with Nash bargaining the weights are exogenous and equal to investors' bargaining powers, while with price posting the weights are endogenous and depend on the queue length of the submarket in which the trade takes place. To better understand the implications of this result, we use (15) to express the gains from a trade between owner δ_i and non-owner δ as

[17] All search and matching models of OTC trading with ex post bargaining feature a version of this equation; for example, it is equation (11) in Duffie et al. (2005).

functions of the match surplus: $p(\delta_i, \delta) - \Delta V(\delta_i, p) = \eta(q)S(\delta_i, \delta)$ and $\Delta V^*(\delta) - p(\delta_i, \delta) = [1 - \eta(q)]S(\delta_i, \delta)$. These expressions correspond to the Hosios (1990) condition for efficiency in markets with search frictions. However, as discussed by Shi (2001) and Eeckhout and Kircher (2010a), in environments with two-sided heterogeneity, equation (15) is necessary but not sufficient for the equilibrium to be efficient.[18] We show that the equilibrium in our model is actually efficient in Section 3.5.

Our results on the price and surplus along the equilibrium path allow us to show that in equilibrium agents will endogenously segment into different submarkets. That is, every submarket features a unique pair of seller and buyer types. Intuitively, price competition among sellers drives the price to the weighted sum of reservation utilities of owners and non-owners that trade in the submarket. Hence, if there are two buyer types in a submarket, one of them does not receive their market utility (since these are strictly increasing in investors' types). This contradicts optimal behavior of non-owners. Symmetrically, when two sellers of different types post the same price, there is a profitable deviation for one of them.

Proposition 1. *Every submarket in equilibrium features only one type of owner and one type of non-owner. Furthermore, if owner δ_i and non-owner δ_j trade in equilibrium, they trade in only one submarket.*

As a consequence, we can index submarkets by (δ_i, δ_j), where δ_i is the owner type and δ_j is the non-owner type in this submarket. Accordingly, we write $q(\delta_i, \delta_j)$ and $p(\delta_i, \delta_j)$ for submarket (δ_i, δ_j). We use the star superscript to denote variables along the equilibrium path: $q^*(\delta_i, \delta_j)$ and $p^*(\delta_i, \delta_j)$.

We should emphasize that focusing on price posting instead of allowing owners to post more complicated contracts is without any loss of generality, given the matching protocol

[18] Condition (15) dictates how a given pair (δ_i, δ) should split the match surplus to achieve efficiency. It does not specify *which pairs* (δ_i, δ) should trade in equilibrium. That is, with two-sided heterogeneity the question of which pairs are formed in equilibrium is crucial for efficiency; this question is trivial in models with homogeneous agents or with one-sided heterogeneity.

we use. As shown by Eeckhout and Kircher (2010b), in the case of bilateral meetings it is suboptimal for sellers to use more complicated mechanisms than price posting to attract buyers. In the case of complete information, the following intuition for this result is straightforward. Consider an owner contemplating whether to post a single price that attracts only non-owners δ_i versus a contract targeted towards non-owners δ_i and δ_j, with $\delta_i < \delta_j$, according to which she transfers the asset to either δ_i or δ_j with order execution rates p_i and p_j respectively. Since the surplus when matching with non-owners δ_i is greater, the owner has greater return when she trades with type δ_i than δ_j. Hence, conditional on meeting a non-owner, the owner receives greater returns in a submarket in which she offers the asset to buyers of type δ_i only.[19] For the owner to find posting a contract weakly better than a single price, it must be the case that her order arrival rate is larger in the submarket with both non-owners δ_i and δ_j than, in the market with non-owners δ_i only. But this means that non-owners of type δ_i have a strictly lower order execution rates in the submarket where both types of non-owners participate. Hence, non-owners δ_i would not show up to trade in the submarket with both types of non-owners, which shows that posting a contract does not make sellers better off.[20]

3.3 Intermediation

Having established that the equilibrium consists of a set of submarkets populated by pairs of owners and non-owners, a natural next step is to look at the pattern of trade by examining the set of submarkets which open in equilibrium.

Proposition 2. *In equilibrium, no market (δ_i, δ_j) with $i > j$ opens. Furthermore, if submarket (δ_i, δ_j) with $i < j$ opens, then markets (δ_i, δ_i) and (δ_j, δ_j) do not open.*

[19] In any equilibrium the prices for all trades have to satisfy our equation (15). If they do not, then there are profitable deviations for both buyers and sellers as a result of non-owners receiving their outside options and owners competing for buyers.

[20] The exact same argument applies if we consider contracts that attract more than two types of non-owners or contracts that feature a lottery.

As expected, pairs with negative surplus ($i > j$) do not trade in equilibrium. Moreover, the only occasion in which zero-surplus pairs of investors ($i = j$) occurs is when this type does not participate in any submarket with positive surplus. If there is no opportunity for trades with positive surplus, investors are indifferent between non-participation and trading with investors with the same valuation. In such a situation we assume that when owners are indifferent between being idle and participating in some submarket, they post a price which attracts a positive queue length. Given the off equilibrium beliefs, this implies that if there are both owners and non-owners of some type i that do not participate in any market with a positive surplus they participate in (δ_i, δ_i). Furthermore, the following corollary follows immediately.

Corollary 1. *If some owners (non-owners) of a given type δ_i are sellers (buyers) on some submarket, then all owners (non-owners) of that type are sellers (buyers) on some submarket.*

Proof. The proof is immediate from proposition 2. ■

Of course in equilibrium not all submarkets with positive surplus necessarily open. In general answering the question of which submarkets open depends on parameter values and on the functional form of the matching function. Instead, we retain our general framework and focus on participation, i.e. which owners are going to be active sellers and which non-owners would be active buyers. This approach allows us to explore the existence of intermediation in our model without making restrictive parametric assumptions.[21]

The potential benefit to investors from participating in a submarket is affected by their reservation value. When a non-owner buys the asset she pays the price but receives her reservation value. Since equilibrium reservation values are strictly increasing in the investor's type it follows that higher type non-owners have a stronger inventive to participate in some submarket as compared to lower types. Similarly, when an owner sells the asset she loses her

[21] As we show later in this section, the qualitative features of the equilibrium pattern of trade depend on the asymptotic properties of the matching function.

reservation value. As a consequence, the higher the type of the owner the lower her incentive to participate in some submarket. The next proposition formalizes this intuition.

Proposition 3. *In equilibrium, an owner type i is an active seller if and only if $i \leq s$ and a non-owner type j is an active buyer if and only if $j \geq b$, where s and b are some types between* 1 *and* I.

In equilibrium, all non-owners of high enough valuation are active buyers and all owners of low enough valuation are active sellers. Intermediation would arise if $s \geq b$. In that situation, there would be some types of investors who are actively selling the asset when they have it, and actively trying to acquire the asset when they do not have it. Whether or not this is true, however, depends on the primitives of the model, and in particular, on the properties of the matching function.

Intuitively, take some agents i, j, k with $\delta_i < \delta_j < \delta_k$. Among these 3 types of agents, the largest gains from trade are generated when an owner type i sells the asset to a non-owner type k. For intermediation to occur, it must be the case that the owner type i and non-owner type k both get favorable terms of trade when they trade with investors type j, even though the surplus generated in these trades is relatively smaller. Thus, it must be the case that i (k) extracts a large fraction of the surplus when she trades with non-owner (owner) type j, or that she can execute a trade relatively fast. Otherwise, owner type i (non-owner type k) would be better off trading with non-owners type k (owners type i). On the other hand, investors type j always have an incentive to act as intermediaries. If they do not intermediate, then they are either idle owners or idle non-owners. However, if they are active, there is the chance to execute a positive surplus trade and extract some of that surplus. Thus, intermediation in our model arises when there exist some intermediate types j that can provide favorable enough terms of trade and fast order execution speeds when they are both owners and non-owners.

Next, we investigate the conditions under which a given non-owner type j accepts terms of trade favourable to an owner type $i < j$ so that j is not an idle non-owner. To this end, consider some non-owner of type j and suppose that she is idle in equilibrium. Given our previous results, this means that all non-owners type j are idle. This implies that $V_0^*(\delta_j) = \gamma V_0^*/(r+\gamma)$. Then consider a deviation by some owner type $i < j$. Given that non-owners j are idle, the deviant can post a price that extracts all of the surplus and still expect a positive queue length. Then the payoff of the deviant is given by

$$V_1(\delta_i|\delta_j) = \frac{\delta_i + \gamma V_1^*}{r+\gamma} + \frac{\lambda(q)}{r+\gamma} S(\delta_i, \delta_j), \tag{16}$$

where q is the anticipated queue length and $V_1(\delta_i|\delta_j)$ is the utility of the deviant and the trade surplus is then given by $S(\delta_i, \delta_j) = \Delta V^*(\delta_j) - \Delta V(\delta_i|\delta_j)$, with $\Delta V(\delta_i|\delta_j) = V_1(\delta_i|\delta_j) - V_0^*(\delta_i)$ being the reservation value for the deviant seller. Using (16), we can express the surplus as

$$\begin{aligned} S(\delta_i, \delta_j) &= \frac{r+\gamma}{r+\gamma+\lambda(q)} \left[\Delta V^*(\delta_j) + V_0^*(\delta_i) - \frac{\delta_i + \gamma V_1^*}{r+\gamma}\right] \\ &= \frac{r+\gamma}{r+\gamma+\lambda(q)} \left[\Delta V^*(\delta_j) - \Delta V^*(\delta_i) + \frac{\lambda(q^*(\delta_i,\delta_k))\eta(q^*(\delta_i,\delta_k))}{r+\gamma} S^*(\delta_i,\delta_k)\right], \end{aligned} \tag{17}$$

where the second line follows from plugging in the solution for the price, (15), into the expression for the owner's value function along the equilibrium path, (10), and $k > j$ is a non-owner type with whom owners type i trade in equilibrium.[22] Then, the following equation expresses the difference in the reservation values of the owner type i when she

[22] By Proposition 2 if k is an active non-owner, then all non-owners with type higher than hers are active as well. Thus, if $j \geq k$ the result that non-owners j are not idle in equilibrium would follow immediately.

posts the deviation price and when she follows her equilibrium strategy:

$$\Delta V(\delta_i|\delta_j) - \Delta V^*(\delta_i) = \frac{1}{r+\gamma+\lambda(q)}\left[\lambda(q)S^*(\delta_i,\delta_j) - \lambda(q^*(\delta_i,\delta_k))\eta(q^*(\delta_i,\delta_k))S^*(\delta_i,\delta_k)\right] \tag{18}$$

The left hand side of (18) is just another way of expressing $V_1(\delta_i|\delta_j) - V_1^*(\delta_i)$ and thus captures the net benefit from deviating. On the right-hand side of the equation we see the decomposition of this net benefit. It is a weighted average of two terms: (i) the payoff from deviating and (ii) the opportunity cost. The benefit from deviating is that the owner has the opportunity to trade with non-owner j. If this happens, she receives all of the surplus, so her gain is $S^*(\delta_i, \delta_j)$. The rate with which she receives an order is $\lambda(q)$, so her expected gain is $\lambda(q)S^*(\delta_i, \delta_j)$. The opportunity cost of deviating is missing out on the search option along the equilibrium path, which is the second term. In equilibrium, the owner trades with a buyer type k and receives a fraction $\eta(q^*(\delta_i, \delta_k))$ of the surplus. This order arrives at the rate $\lambda(q^*(\delta_i, \delta_k))$. Thus, while the owner i is waiting for an order from an investor type j, she is missing out on a potential order from her equilibrium trading counter-party type k. As a consequence, the owner type i would have an incentive to deviate when the fraction of the surplus she receives along the equilibrium path is too low, or if deviating provides a quick trading opportunity. Observe that given the out-of-equilibrium beliefs, the deviant expects a queue length $q = \infty$, and so the order arrival rate would depend on the behavior of the matching function in the limit. If it satisfies the Inada-type condition $\lim_{q\to\infty} \lambda(q) = \infty$, then the net benefit from deviating converges to $S^*(\delta_i, \delta_j) > 0$. Since this is a strictly profitable deviation it must be that non-owners type j are not idle in equilibrium.

Intuitively, if $\lambda(q)$ approaches infinity, the owner type i can transfer the asset to a non-owner type j immediately. Once this happens, the asset is held by an agent with a higher utility flow. The investor type j can then post the asset for sale, offering the same terms that investors i offer to non-owners type k in equilibrium. Thus, she executes the exact same orders that the type i owner would execute in equilibrium, but she enjoys a higher utility flow while waiting for an order. Since the deviant investor type i extracts all of the surplus, she can post a price that transfers all of the potential gains of having a higher utility flow while waiting for an order from non-owners type k. Thus, the deviant essentially trades her reservation value for that of an investor type j. Moreover, since $\lambda(q) \to \infty$, she can do the exchange instantaneously and so there is no opportunity cost of posting the deviation price. As a result, the deviant extracts all of the surplus from the trade with an investor type j and receives an order immediately. Since this is true for all types $i < j$, it follows that no non-owner type $j \geq 2$ is idle in equilibrium.

Similarly, we can investigate the conditions under which a given owner type j can offer favourable enough terms of trade to non-owners type $k > j$, so that the owner is not idle in equilibrium. To this end, consider an idle owner who contemplates a deviation by posting a price that is aimed at attracting some type k. The best chance she has of inducing a positive queue length is if she posts a price that leaves all of the surplus to the buyer. Given such a deviation, a non-owner type k who participates receives a utility given by

$$V_0(\delta_k|\delta_j) = \frac{\gamma V_0^*}{r+\gamma} + \frac{\lambda(q)/q}{r+\gamma} S(\delta_j, \delta_k), \tag{19}$$

where q is the induced queue length. Analogously to the case when investor j was an idle non-owner, the surplus from the trade is given by

$$S(\delta_j, \delta_k) = \frac{r+\gamma}{r+\gamma+\lambda(q)/q} \left[\Delta V^*(\delta_k) - \Delta V^*(\delta_j) + \frac{\lambda(q^*(\delta_i, \delta_k))[1-\eta(q^*(\delta_i, \delta_k))]}{q^*(\delta_i, \delta_k)(r+\gamma)} S^*(\delta_i, \delta_k) \right], \tag{20}$$

where $i < j$ is an owner type with whom non-owners type k trade in equilibrium.[23] Hence,

[23] By Proposition 2, if i is an active owner, then so are all owners of type less than i. Thus, if $j \leq i$, the

the net benefit of participating in the submarket for a non-owner type k is

$$\Delta V^*(\delta_k) - \Delta V(\delta_k|\delta_j) = \frac{1}{r+\gamma+\lambda(q)/q}\left[\frac{\lambda(q)}{q}S^*(\delta_j,\delta_k) - \frac{\lambda(q^*(\delta_i,\delta_k))}{q^*(\delta_i,\delta_k)}[1-\eta(q^*(\delta_i,\delta_k))]S^*(\delta_i,\delta_k)\right]. \tag{21}$$

The left-hand side of the expression is simply the net benefit from participating in the deviation submarket, $V_0(\delta_k|\delta_j) - V_0^*(\delta_k)$. The right-hand side decomposes the net payoff: the first term is the expected payoff from participating in the submarket, and the second term is the opportunity cost of foregoing a potential order execution in the equilibrium submarket. If the matching function satisfies the Inada-type condition $\lim_{q\to 0}\lambda(q)/q = \infty$, then the net payoff from participating converges to $S^*(\delta_j,\delta_k)$, which is strictly positive. Hence, given the out-of-equilibrium beliefs, a continuity argument implies that the posted deviation price will induce a positive queue length. Since the payoffs for both the owner type j and the non-owner type k are continuous in the price, it follows that a deviation which leaves a small enough fraction of the surplus to the deviant would still induce a positive queue length. Since this is a strictly profitable deviation, it follows that owners type j will not be idle in equilibrium. Since this deviation is profitable for all idle owners type $j<k$, it follows that in equilibrium no owner type $j \leq I-1$ is idle.

Intuitively, the deviation price is just large enough so that the seller can cover her expected costs of reacquiring the asset, which are given by her reservation value, $\Delta V^*(\delta_j)$. Thus, if they queue on the submarket, non-owners type k can acquire the asset immediately at the cost of $\Delta V^*(\delta_j)$. By doing so, they save on the utility cost of not having the asset and searching for it on the submarket (δ_i,δ_k). Since reservation values are strictly increasing in investors' type, the utility loss from not having the asset is larger for investors type k than it is for investors type j, i.e. $\Delta V^*(\delta_k) > \Delta V^*(\delta_j)$. Hence, non-owners type k always have an

result that owners of type j are not idle in equilibrium would follow immediately.

incentive to queue for the deviant's price.

Proposition 4. *If the matching function satisfies the Inada-type conditions* $\lim_{q\to\infty}\lambda(q) = \lim_{q\to 0}\lambda(q)/q = \infty$*, then* $\delta_b = \delta_2$ *and* $\delta_s = \delta_{I-1}$*.*

Proof. The proof is in the preceding text. ∎

It is clear from the preceding text that when the matching function does not satisfy the Inada-type conditions, it is possible that no intermediation takes place. Intuitively, when the expected waiting times for an order execution are bounded away from zero, the potential intermediary may not be able to offer favourable enough terms of trade for any level of the induced queue length on the deviation submarket. Another instance when our conclusion may not hold is when there is a flow cost from participating in a submarket. In that case, the potential intermediary may not be willing to offer favourable terms, because if she offers very fast execution speeds for her trading counter-party, this implies very low matching rates for her. In our benchmark case this is not a problem, because the potential intermediary is idle in equilibrium and so suffers no opportunity cost by waiting a long time for an order execution. With participation costs, however, high waiting times imply a high expected cost of trading and she may prefer to stay idle. Finally, intermediation may not arise in the frictionless limit of the model. As equilibrium trading delays go to zero, idle sellers (buyers) of type j cannot induce a deviation by buyers of type k (sellers of type i) by offering strictly faster trading speeds.

Given the insights from proposition 4, we can show that in equilibrium all types of investors will participate in the market in some capacity, i.e. as either active buyers or sellers. This result holds regardless of whether or not the matching function satisfies the Inada-type conditions.

Corollary 2. *In equilibrium, for any type* i*, either all owners are sellers on some submarket, or all non-owners are buyers on some submarket, or both.*

3.4 Sorting

So far, we have examined the extent of participation in the market and its relation to intermediation. To highlight the linkage between intermediation and sorting, we impose the Inada-type conditions from Proposition 4 on the matching function for the analysis in this section. The next natural step is to understand which non-owner types buy from each owner type. Given that the model features two-sided heterogeneity, this is equivalent to asking: what is the sorting pattern in the model? Sorting has been the topic of an extensive literature, recently surveyed in Chade et al. (2017). Since our model features frictions and directed search, the contributions of Shi (2001), Eeckhout and Kircher (2010a) and Jerez (2014) are particularly relevant.[24]

Following Eeckhout and Kircher (2010a), we define sorting in terms of the contributions of owners and non-owners to the surplus function of the match. This is important because the owners' valuation of the asset enters the surplus function negatively, analogously to the cost of production in a model of the product market. Moreover, and again following Eeckhout and Kircher (2010a), we include only strictly monotone assignments in the definition of assortative matching. Jerez (2014) refers to the strictly increasing assignment as "perfect sorting", a term we also use.

Definition: *We call the equilibrium assignment positive assortative (PAM) if there exists a strictly decreasing sorting function $\mu : \mathcal{I} \to \mathcal{I}$ that assigns a non-owner type δ' to every owner type δ such that submarket (δ, δ') opens in equilibrium. If μ is strictly increasing, we call the assignment negative assortative (NAM).*

Armed with this definition we can show the following result.

Lemma 2. *Suppose the matching function satisfies $\lim_{q\to\infty} \lambda(q) = \lim_{q\to 0} \lambda(q)/q = \infty$ and that $I > 2$. Then, PAM cannot be an equilibrium. Moreover, the only NAM equilibrium is*

[24] To the best of our knowledge, we are the first to draw a parallel between the sorting and OTC literatures. This connection seems natural and we hope it will be pursued further in future research.

a single chain which starts at owner type δ_1 and ends at non-owner type δ_I. That is, all possible submarkets of the type (δ_i, δ_{i+1}) open and no other submarket opens.

The perfect sorting requirement of PAM and NAM imposes a very strict structure regarding trading pairs in equilibrium. In essence, PAM implies that some investor types will not participate in the asset market and precludes intermediation. However, as we have seen, near-complete investor participation and intermediation are essential features of the equilibrium in our model. The only kind of strictly monotone assignment consistent with the equilibrium of our model is the single chain of NAM. Generally, though, in models with two-sided heterogeneity, a given non-owner type may, in principle, trade with more than one owner types — and vice versa. That is, matching between types may well not be strictly monotone. Jerez (2014) provides an example of such an assignment, and in our model, such an imperfect assignment very often arises.[25] This is in sharp contrast with Shi (2001) and Eeckhout and Kircher (2010a), who focus on perfect/strictly monotone assignments.

There are two important model differences that explain why imperfect assignments are ruled out in the equilibria of Shi (2001) and Eeckhout and Kircher (2010a), but may well occur in our model. First, Shi (2001) assumes free entry of agents on one side of the market, while we follow the OTC literature which usually assumes the masses of agents are fixed. In Shi's set up, one side of the market chooses and commits to a type before entering the market, while free entry makes agents indifferent between types. If two different types of buyers match with the same type of seller, it would be impossible to make these buyers indifferent in equilibrium. In our model, however, there is no entry, and the same non-owner type may buy from different owner types, as long as prices and queue lengths in different submarkets

[25] Shimer (2005) provides a labor market example in which identical firms choose to gather applications from different types of workers, an imperfect assignment which he calls "mismatch". In the context of our model, this assignment implies that different types of owners participate in the same submarket, which cannot happen in equilibrium. The reason for this difference lies in the choice of matching function: Shimer (2005) uses a many-to-one urn-ball matching function, while we use a one-to-one matching function, as do Lester et al. (2015) and most of the other papers in the sorting literature with directed search.

make the non-owner type indifferent. Second, Eeckhout and Kircher (2010a) assume that agents' types are drawn from a twice continuously differentiable CDF, whereas in our setting the distribution is discrete. This is important, since when the mass of an investor type is large enough, the probability of meeting these investors in the submarkets they participate grows. This means that this investor type may be able to cover the demand/supply of multiple types on the other side of the market, leading to an imperfect equilibrium assignment.

3.5 Equilibrium Existence and Constrained Efficiency

In this Section, we establish existence and constrained efficiency of the decentralized equilibrium. To do so, we show that the equilibrium of the decentralized model coincides with the solution to the planner's problem. Setting up the planner's problem is straightforward but tedious, so we defer its full presentation to Appendix A. The planner's objective is to maximize the net present sum of the instantaneous payoffs of all owners. She accomplishes this by assigning owners and non-owners as buyers and sellers in different submarkets. We follow Eeckhout and Kircher (2010a) and restrict the planner to assigning a single owner and a single non-owner type to each submarket. We assume further that if the planner finds it optimal for owner type δ_i to trade with non-owner type δ_j, then all such trades take place in a single market. Both of these assumptions are without loss of generality, since the matching function has constant returns to scale and meetings within a submarket are random. The planner essentially moves masses of owners and non-owners around different submarkets to maximize the total surplus of potential matches. When considering these moves, the planner is constrained by the matching function in each submarket, as well as the laws of motion that determine the evolution of investor masses over time. Of course, the planner is also constrained by the appropriate non-negativity and resource constraints.

Analyzing the planner's problem allows us to derive two important results: first, the solution to the planner's problem coincides with the decentralized equilibrium. Second, a solution to the planner's problem (and, as a result, a decentralized equilibrium) exists. This is our Proposition 5 below. Although OTC search models with exogenous contact rates are typically contrained efficient (see, e.g., Hugonnier et al. (2014) or Afonso and Lagos (2015)), this does not trivially generalize in the case of endogenous contact rates. Specifically, Lagos and Rocheteau (2007) and Lester et al. (2015) show that in a model with random search, ex post bargaining and contact rates given by a matching function, the equilibrium is typically inefficient, unless a version of the Hosios (1990) condition holds. The equilibrium is also inefficient in the model of Farboodi et al. (2017b), in which contact rates are the result of ex-ante identical agents choosing how much to invest in the quality of their search technology. These inefficiencies arise due to externalities in the matching process and ex post bargaining between investors. Following the directed search literature, we show that when owners post prices and non-owners choose which price to search for, these externalities are internalized through this pricing mechanism. In this sense, our Proposition 5 generalizes the results of Lester et al. (2015) for a fully decentralized market.

Proposition 5. *The decentralized equilibrium exists and it is constrained efficient.*

4 Conclusion

In this paper, we have built a search-theoretic model of a fully decentralized OTC market with complete price transparency. To capture price transparency, we use the competitive search protocol: asset owners, who are heterogeneous with respect to their asset valuation, post prices that are available to potential buyers; buyers, who are also heterogeneous in their asset valuations, observe all prices and decide to which seller they will direct their order. Investors self-segment into different submarkets, in which they trade the asset at different prices. The speed of trade execution in each submarket is endogenous and depends on the ratio of sellers to buyers willing to trade at the sellers' posted price. Prices reflect the different valuations of investors and allocate the orders of buyers into different submarkets efficiently. That is, the equilibrium of the asset market is constrained efficient, a typical property of competitive search.

The main result of the paper is that in competitive search, equilibrium intermediation emerges endogenously: some agents choose to act as dealers, selling the asset when they have it and buying the asset when they do not have it. Moreover, intermediation in the model is extensive: for a broad class of matching functions, all agents, except the lowest and highest valuation ones, choose to intermediate trades. Since our model features a two-sided market with heterogeneous agents choosing their preferred counterparty, characterizing the trade pattern boils down to a sorting problem. In contrast with the sorting literature, we find that imperfect assignments may occur naturally in our OTC framework.

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A Planner's problem and model solution

The complete planner's problem is presented below. All variables are functions of time, but we do not make that explicit in the notation to keep it succinct.

$$\max_{\{o(\delta_i,\delta_j)\}_{i,j\in\mathcal{I}},\{n(\delta_i,\delta_j)\}_{i,j\in\mathcal{I}}\in U} \int_0^\infty exp(-rt) \sum_i o(\delta_i)\delta_i dt \tag{22}$$

s.t.

$$\begin{aligned} \dot{o(\delta_i)} = \gamma f(\delta_i) \sum_k o(\delta_k) - \gamma o(\delta_i) \sum_k f(\delta_k) + \sum_k m\left(o(\delta_k,\delta_i), n(\delta_k,\delta_i)\right) \\ - \sum_j m\left(o(\delta_i,\delta_j), n(\delta_i,\delta_j)\right), \quad \forall i \end{aligned} \tag{23}$$

$$\begin{aligned} \dot{n(\delta_j)} = \gamma f(\delta_j) \sum_k n(\delta_k) - \gamma n(\delta_j) \sum_k f(\delta_k) + \sum_k m\left(o(\delta_j,\delta_k), n(\delta_j,\delta_k)\right) \\ - \sum_i m\left(o(\delta_i,\delta_j), n(\delta_i,\delta_j)\right), \quad \forall j \end{aligned} \tag{24}$$

$$o(\delta_i) - \sum_j o(\delta_i,\delta_j) \geq 0, \quad \forall i \tag{25}$$

$$n(\delta_j) - \sum_i n(\delta_i,\delta_j) \geq 0, \quad \forall j \tag{26}$$

$$o(\delta_i,\delta_j) \geq 0, \quad \forall i,j \tag{27}$$

$$n(\delta_i,\delta_j) \geq 0, \quad \forall i,j \tag{28}$$

$$\lim_{t\to\infty} o(\delta_i) \geq 0, \quad \lim_{t\to\infty} n(\delta_i) \geq 0, \quad \forall i \tag{29}$$

$$o(\delta_i),\ n(\delta_i) \text{ given at time } t=0, \quad \forall i \tag{30}$$

where $U = [0,1]^{I\times I\times 2}$ is the range over which the planner optimizes the masses of owners and non-owners.

B Proofs

Lemma 1. *For any $i > j$, the following inequalities hold: $\Delta V^*(\delta_i) > \Delta V^*(\delta_j)$; $V_0^*(\delta_i) \geq V_0^*(\delta_j)$; $V_1^*(\delta_i) > V_1^*(\delta_j)$. Furthermore, if some non-owners of type j are active in some* *submarket, then* $V_0^*(\delta_i) > V_0^*(\delta_j)$.

Proof. We prove the statements in the order the appear in the text. First, let $(p^{'}(\delta_i), q^{'}(\delta_i))$ be an optimal price and queue length for an owner type $o(\delta_i)$. Analogously, let $(p^{''}(\delta_i), q^{''}(\delta_i))$ be the price and queue length on a market that non-owner type $n(\delta_i)$ optimally chooses to visit. Then,

$$(r+\gamma)V_1^*(\delta_i) = \delta_i + \gamma V_1^* + \lambda(q^{'}(\delta_i)) \left[p^{'}(\delta_i) - \Delta V^*(\delta_i)\right], \tag{31}$$

$$(r+\gamma)V_0^*(\delta_i) = \gamma V_0^* + \frac{\lambda(q^{''}(\delta_i))}{q^{''}(\delta_i)} \left[\Delta V^*(\delta_i) - p^{''}(\delta_i)\right]. \tag{32}$$

Now, to the contrary, suppose that there exist some types δ_i, δ_j, with $i > j$, such that $\Delta V^*(\delta_i) \leq \Delta V^*(\delta_j)$. Then,

$$\begin{aligned}(r+\gamma)V_0^*(\delta_j) &\geq \gamma V_0^* + \frac{\lambda(q^{''}(\delta_i))}{q^{''}(\delta_i)} \left[\Delta V^*(\delta_j) - p^{''}(\delta_i)\right] \geq \\ &\geq \gamma V_0^* + \frac{\lambda(q^{''}(\delta_i))}{q^{''}(\delta_i)} \left[\Delta V^*(\delta_i) - p^{''}(\delta_i)\right] = (r+\gamma)V_0^*(\delta_i)\end{aligned}$$

where the first inequality follows by the optimality of $\left(p^{''}(\delta_i), q^{''}(\delta_i)\right)$ and the second one by assumption. Hence, $V_0^*(\delta_j) \geq V_0^*(\delta_i)$. Furthermore, $V_1^*(\delta_i) > V_1^*(\delta_j)$, since

$$\begin{aligned}(r+\gamma)V_1^*(\delta_i) &\geq \delta_i + \gamma V_1^* + \lambda(q^{'}(\delta_j)) \left[p^{'}(\delta_j) - \Delta V^*(\delta_j)\right] > \\ &> \delta_j + \gamma V_1^* + \lambda(q^{'}(\delta_j)) \left[p^{'}(\delta_j) - \Delta V^*(\delta_j)\right] = (r+\gamma)V_1^*(\delta_j)\end{aligned}$$

But this implies that $\Delta V^*(\delta_i) > \Delta V^*(\delta_j)$ which is a contradiction.

Next, to prove the second statement

$$
\begin{aligned}
(r+\gamma)V_0^*(\delta_i) \geq \gamma V_0^* + \frac{\lambda(q^{''}(\delta_j))}{q^{''}(\delta_j)}\left[\Delta V^*(\delta_i) - p^{''}(\delta_j)\right] \geq \\
\geq \gamma V_0^* + \frac{\lambda(q^{''}(\delta_j))}{q^{''}(\delta_j)}\left[\Delta V^*(\delta_j) - p^{''}(\delta_j)\right] = (r+\gamma)V_0^*(\delta_j)
\end{aligned}
$$

The third statement follows, since $\Delta V^*(\delta_i) = V_1^*(\delta_i) - V_0^*(\delta_i) > V_1^*(\delta_j) - V_0^*(\delta_j) = \Delta V^*(\delta_j)$. Hence, $V_1^*(\delta_i) - V_1^*(\delta_j) > V_0^*(\delta_i) - V_0^*(\delta_j) \geq 0$ and $V_1^*(\delta_i) > V_1^*(\delta_j)$.

■

Proposition 1. *Every submarket in equilibrium features only one type of owner and one type of non-owner. Furthermore, if owner δ_i and non-owner δ_j trade in equilibrium, they trade in only one submarket.*

Proof. Consider the owner's problem given by (9). Use equations (10) and (11) along with the constraint $V_0(\delta, p) = V_0^*(\delta)$ to substitute out for the price. Then the owner's problem can be rewritten as

$$
\max_{q,\delta} \left\{\lambda(q)S(\delta_i, \delta) - q\left[(r+\gamma)V_0^*(\delta) - \gamma V_0^*\right]\right\} \tag{33}
$$

where $S(\delta_i, \delta) = \Delta V^*(\delta) - V_1(\delta_i, p) + V_0^*(\delta_i) = \Delta V^*(\delta) - \Delta V(\delta_i)$. Thus, each owner type chooses which non-owner types she wishes to attract and at what queue length.

We proceed by introducing three lemmas. First, we show that if a particular owner type δ_i wishes to attract any two distinct non-owner types δ_i' and δ_i'', she will find it optimal to do so in different submarkets. Second, we show that if an owner type δ_i wishes to attract a particular non-owner type δ_i', there is a unique pay-off maximizing pair (p, q). Lastly, we show that no two owner types find it optimal to post the same price in equilibrium.

Lemma 3. *If an owner type δ_i wishes to attract non-owners δ_i' and δ_i'', with $\delta_i' \neq \delta_i''$, she finds it optimal to do so at different prices.*

Proof. The first order condition with respect to the queue length is given by

$$\lambda'(q)S(\delta_i, \delta) = (r + \gamma)V_0^*(\delta) - \gamma V_0^* \tag{34}$$

which is equivalent to

$$\lambda'(q)S(\delta_i, \delta) = \frac{\lambda(q)}{q}\left[\Delta V^*(\delta) - p\right] \tag{35}$$

Hence, if $\varepsilon(q) \equiv \frac{\lambda'(q)q}{\lambda(q)}$ is the elasticity of the owner meeting rate $\lambda(q)$ with respect to the queue length, it follows that

$$p = [1 - \varepsilon(q)]\Delta V^*(\delta) + \varepsilon(q)\left[V_1(\delta_i, p) - V_0^*(\delta_i)\right] \tag{36}$$

Notice that there is a direct relationship between $\varepsilon(q)$ and $\eta(q)$: $\eta(q) = 1 - \varepsilon(q)$, since $\lambda(q) = m(1, q)$. But by Lemma 1 the above can hold for at most one type of non-owner δ, because ΔV^* is strictly increasing. This proves the lemma.

■

Lemma 4. *All owners type δ_i wishing to attract non-owner type δ post the same price.*

Proof. By equation (36), the price is determined uniquely by the reservation values of the owner and non-owner which then implies that the price is determined by the queue length. Thus, conditional on attracting non-owner type δ, if owner δ_i finds it optimal to induce a unique queue length, the lemma is proven; i.e. we need to show that equation (34) has a unique solution for q. Consider owner type δ_i who wishes to attract non-owner of type δ. Substituting the first order condition for the price, (36), into the value functions for the owner and the non-owner, yields

$$(r+\gamma)V_1(\delta_i) = \delta_i + \gamma V_1^* + \lambda(q)\,(1-\varepsilon(q))\,S(\delta_i,\delta), \tag{37}$$

$$(r+\gamma)V_0(\delta) = \gamma V_0^* + \frac{\lambda(q)}{q}\varepsilon(q)S(\delta_i,\delta). \tag{38}$$

Hence,

$$(r+\gamma)S(\delta_i,\delta) = (r+\gamma)\Delta V^*(\delta) - \delta_i - \gamma V_1^* - \lambda(q)\,(1-\varepsilon(q))\,S(\delta_i,\delta) + (r+\gamma)V_0^*(\delta_i) \tag{39}$$

$$\Rightarrow S(\delta_i,\delta) = \frac{(r+\gamma)\,[\Delta V^*(\delta) + V_0^*(\delta_i)] - \delta - \gamma V_1^*}{r+\gamma+\lambda(q)\,(1-\varepsilon(q))} \tag{40}$$

By the concavity of $\lambda(q)$, the expression $\lambda(q)\,(1-\varepsilon(q)) = \lambda(q) - \lambda'(q)q$ is strictly increasing. Thus, both surplus and $\lambda'(q)$ are strictly decreasing in q, which then proves the lemma.

■

Thus, lemmas 3 and 4 imply that the owner's problem reduces to choosing an optimal pair (q,p) subject to attracting a non-owner type δ and then choosing the non-owner type δ which maximizes her payoff.

Lemma 5. *No two owners δ_i and δ_j, with $\delta_i \neq \delta_j$, find it optimal to post the same price given that this price attracts a positive queue length.*

Proof. Suppose not. Then there exist two owners of type δ_i and δ_j, with $\delta_i \neq \delta_j$, that find it optimal to post the same price p which attracts a positive queue length of non-owners, q. By Lemma 3 each owner type has a particular non-owner type they wish to attract, say δ_i' and δ_j' respectively. Since both owner types post the same price, the market is populated by both types of owners and both types of non-owners. Thus owner type δ_i expects to trade with positive probability with both non-owner types δ_i' and δ_j'. Now consider a deviation by an owner type δ_i who wishes to attract δ_j' on some submarket. By lemmas 3 and 4 the unique optimal price for such a deviation is $p_j \neq p$, which induces a queue length $q_j \neq q$. Since the original submarket (p, q) was a feasible option, the deviant would find it strictly better to trade with non-owner type δ'_j on a submarket p_j than on submarket p. Furthermore, market p is populated by non-owner types δ'_i and δ'_j, so the payoff of owner δ_i on market p is the same whether she trades with type δ'_i or δ'_j. Hence, owners type δ_i posting a price p with the aim of attracting non-owners type δ'_i have a strictly profitable deviation by posting a price p_j and attracting non-owner type δ'_j. Hence, price p cannot be an optimally set price for owner type δ_i. ■

This concludes the proof of Proposition 1.

■

Proposition 2. *In equilibrium, no market (δ_i, δ_j) with $i > j$ opens. Furthermore, if submarket (δ_i, δ_j) with $i < j$ opens, then markets (δ_i, δ_i) and (δ_j, δ_j) do not open.*

Proof. First, we prove the statement for $i > j$. By Lemma 1 if $i > j \Rightarrow \Delta V^*(\delta_i) > \Delta V^*(\delta_j)$. Hence, $S^*(\delta_i, \delta_j) = \Delta V^*(\delta_j) - \Delta V^*(\delta_i) < 0$. This then contradicts the optimality of (p, q) on that market since the owner would be strictly better off by posting a price that does not attract any non-owners.

Second, we prove the statement for $i < j$. Since market (δ_i, δ_j) opens, it follows that $S^*(\delta_i, \delta_j) > 0$. Suppose to the contrary that (δ_j, δ_j) opens. Then, $V_0(\delta_i, \delta_j) = V_0(\delta_j, \delta_j) = V_0^*(\delta_j)$. But since $S^*(\delta_j, \delta_j) = 0$, it follows that $\lambda' \left(q(\delta_i, \delta_j)\right) S^*(\delta_i, \delta_j) = 0$ by equations (11) and (36). But this is a contradiction since $\Delta V^*(\delta_j) > \Delta V^*(\delta_i)$ by Lemma 1 and in equilibrium $q(\delta_i, \delta_j) = n(\delta_i, \delta_j)/o(\delta_i, \delta_j) < \infty$.

Analogously, suppose that market (δ_i, δ_i) opens. Thus, equations (10) and (36) imply that $[1 - \varepsilon\left(q(\delta_i, \delta_i)\right)] \lambda\left(q(\delta_i, \delta_i)\right) S^*(\delta_i, \delta_i) = [1 - \varepsilon\left(q(\delta_i, \delta_j)\right)] \lambda\left(q(\delta_i, \delta_j)\right) S^*(\delta_i, \delta_j)$, which is a contradiction since $S^*(\delta_i, \delta_i) = 0$ and $q(\delta_i, \delta_i) < \infty$ in equilibrium. ■

Proposition 3. *In equilibrium, an owner type i is an active seller if and only if $i \leq s$ and a non-owner type j is an active buyer if and only if $j \geq b$, where s and b are some types between 1 and I.*

Proof. First we show that an owner type δ_i is an active seller if and only if $\delta_i \leq \delta_s$. Since there are potential gains from trade and there are no participation costs, it is easy to see that at least one market would open in equilibrium. Then, let δ_s be the highest type among any active sellers in equilibrium. We will show that any owner $\delta_i \leq \delta_s$ is an active seller as well. Suppose to the contrary that there exists some owner type $\delta_i \leq \delta_s$ that is not an active seller. Let δ' denote a non-owner type that buys from the active seller type δ_s, then

$$(r+\gamma)V_1^*(\delta_s) = \delta_s + \gamma V_1^* + \lambda(q(\delta_s,\delta'))\left[p(\delta_s,\delta') - \Delta V^*(\delta_s)\right], \tag{41}$$

$$(r+\gamma)V_1^*(\delta_i) = \delta_i + \gamma V_1^*. \tag{42}$$

Since $p(\delta_s,\delta') - \Delta V^*(\delta_s) > 0$ and $\Delta V^*(\delta_s) \geq \Delta V^*(\delta_i)$, this means that the owner type δ_i has a strictly profitable deviation by posting the price $p(\delta_s,\delta')$ and attracting length $q(\delta_s,\delta')$.

Similarly, let δ_b be the lowest type among any active buyers in equilibrium. Suppose to the contrary that some non-owner type $\delta_j \geq \delta_b$ is not an active buyer. If the non-owner δ_b trades with some owner δ in equilibrium, then

$$(r+\gamma)V_0^*(\delta_b) = \gamma V_1^* + \frac{\lambda(q(\delta,\delta_b))}{q(\delta,\delta_b)}\left[\Delta V^*(\delta_b) - p(\delta,\delta_b)\right], \tag{43}$$

$$(r+\gamma)V_0^*(\delta_j) = \gamma V_1^*. \tag{44}$$

Since $\Delta V^*(\delta_b) - p(\delta,\delta_b) > 0$ and $\Delta V^*(\delta_j) \geq V^*(\delta_b)$, the non-owner type δ_j has a strictly profitable deviation by queuing on the market (δ,δ_s).

■

Corollary 2. *In equilibrium, for any type i, either all owners are sellers on some submarket,* *or all non-owners are buyers on some submarket, or both.*

Proof. Suppose to the contrary, that there exist some type j such that owners and non-owners of this type are both idle in equilibrium. Given proposition 3, it must be the case that $1 < j < I$. Then, consider a market (δ_i, δ_k) which opens in equilibrium with $1 \leq i < j < k \leq I$. Then, the net benefit of a seller type i who considers deviating and posting a price that attracts buyers type j, $\Delta V(\delta_i|\delta_j) - \Delta V^*(\delta_i)$, is given by equation (18). Since this is not a profitable deviation, it must be the case that the net gain is non-positive. Furthermore, since the deviation induces a queue length $q = \infty$ and the deviation payoff is strictly increasing in q, at $q = q^*(\delta_i, \delta_k) < \infty$ it follows that

$$\frac{1}{r + \gamma + \lambda(q^*(\delta_i, \delta_k))} [\lambda(q^*(\delta_i, \delta_k))S^*(\delta_i, \delta_j) - \lambda(q^*(\delta_i, \delta_k))\eta(q^*(\delta_i, \delta_k))S^*(\delta_i, \delta_k)] < 0$$

$$\Rightarrow S^*(\delta_i, \delta_j) < \eta(q^*(\delta_i, \delta_k))S^*(\delta_i, \delta_k). \quad (45)$$

Similarly, owners of type j are idle in equilibrium which implies that they cannot offer favorable enough terms of trade to non-owners type k. Hence, posting a deviation price which gives non-owners of type k the whole surplus leaves non-owners of type j with a non-positive net benefit, $\Delta V^*(\delta_k) - \Delta V(\delta_k|\delta_j)$, for any positive queue length. Since $q^*(\delta_i, \delta_k) > 0$, equation (21) implies that

$$S^*(\delta_j, \delta_k) < [1 - \eta(q^*(\delta_i, \delta_k))]S^*(\delta_i, \delta_k). \quad (46)$$

But, then equations (45) and (46) imply that

$$S^*(\delta_i, \delta_k) = S^*(\delta_i, \delta_j) + S^*(\delta_j, \delta_k) < \eta(q^*(\delta_i, \delta_k))S^*(\delta_i, \delta_k) + [1 - \eta(q^*(\delta_i, \delta_k))]S^*(\delta_i, \delta_k) = S^*(\delta_i, \delta_k), \quad (47)$$

which is clearly a contradiction.

■

Lemma 2. *Suppose the matching function satisfies* $\lim_{q\to\infty}\lambda(q) = \lim_{q\to 0}\lambda(q)/q = \infty$ *and that* $I > 2$. *Then, PAM cannot be an equilibrium. Moreover, the only NAM equilibrium is a single chain which starts at owner type* δ_1 *and ends at non-owner type* δ_I. *That is, all possible submarkets of the type* (δ_i, δ_{i+1}) *open and no other submarket opens.*

Proof. Let us start with the PAM statement. Consider first the case that $I = 3$. Then PAM implies that the only submarket which opens is (δ_1, δ_3). But then owner type δ_2 is an idle seller and this is a contradiction. Next, suppose that there are at least 4 types. By Proposition 4 owner types δ_1 and δ_2 are active sellers and non-owner types δ_I and δ_{I-1} are active buyers. Suppose to the contrary that the equilibrium assignment is PAM. Then, submarkets (δ_1, δ_I) and (δ_2, δ_{I-1}) must open. But then, it follows that non-owner type δ_2 is an idle non-owner which is a contradiction.

We continue with the NAM statement. First, in any NAM equilibrium a submarket (δ_i, δ_j) with $j \neq i+1$ never opens. For $j \leq i$ the result is immediate from proposition 3. Next suppose that some submarket (δ_i, δ_j) opens with $j > i+1$. By Proposition 4 and the definition of NAM, it must be the case that each owner type, except I, sells to exactly one non-owner type and each non-owner type, except 1, buys from exactly one owner type. Hence, there are exactly $i-1$ submarkets where owners of type less than i participate. At the same time, it must be the case that there are exactly $j-2 > i-1$ non-owner types with valuations less than j that participate in these submarkets.But this is a contradiction.

Next, we need to show that all markets of the type (δ_i, δ_{i+1}) open. Suppose not, then there exist some i such that the above submarket does not open. But we know that no owner type $j \neq i$ will sell to a non-owner type $i+1$ by the first part of the proof. Thus, $i+1$ must be an idle buyer which is a contradiction. ■

Proposition 5. *The decentralized equilibrium exists and it is constrained efficient.*

Proof. **Existence.** First we show that a solution of the planner's problem exists. To this end, we will invoke Theorem 21 in Seierstad and Sydsaeter (1986) on p.406. To show that our problem satisfies the conditions in the theorem, we translate our problem into their notation and then show each of the 5 conditions hold. First, let $x := (\{o(\delta_i)\}_{i\in\mathcal{I}}, \{n(\delta_j)\}_{j\in\mathcal{I}})$ be the vector of state variables, $u := (\{o(\delta_i, \delta_j)\}_{i,j\in\mathcal{I}}, \{n(\delta_i, \delta_j)\}_{i,j\in\mathcal{I}})$ be the vector of market allocations, $f_0(x, u, t) := exp(-rt) \sum_i o(\delta_i)\delta_i$ be the maximand, $f(x, u)$ be a vector function such that for $i \in [1, I]$, $f_i(x, u)$ is the right hand side of equation (23) for owner type i and for $j \in [I + 1, 2I]$, $f_j(x, u)$ is the right hand side of equation (24) for non-owner type j. Similarly, define $g(x, u)$ to be the left hand side of the constraints (25), (26).

Claim 1. The functions f_0, f, and g are continuos and U is closed. This is obvious.

Claim 2. Define $N(x, U, t) = \{(f_0(x, u, t) + \tilde{\gamma}, f(x, u)) : \tilde{\gamma} \leq 0, g_i(x, u, t) \geq 0, \forall i, u \in U\}$. We will show that this set is convex for all $(x, t) \in \mathbb{R}^{2I} \times [0, \infty)$. Pick any (x, t) and observe that the set $N(x, U, t)$ is all pairs $(f_0(x, u, t) + \tilde{\gamma}, f(x, u))$ induced by some $\tilde{\gamma} \leq 0$ and some $u \in U$ for which the constraints (25), (26) are satisfied. Then, let $(f_0(x, u', t) + \tilde{\gamma}', f(x, u'))$ and $(f_0(x, u'', t) + \tilde{\gamma}', f(x, u''))$ be some elements of $N(x, U, t)$ and let $\lambda \in (0, 1)$. Then, notice that $\lambda(f_0(x, u', t) + \tilde{\gamma}') + (1 - \lambda)(f_0(x, u'', t) + \tilde{\gamma}') = f_0(x, u', t) + \lambda\tilde{\gamma}' + (1 - \lambda)\tilde{\gamma}''$, with $\lambda\tilde{\gamma}' + (1 - \lambda)\tilde{\gamma}'' \leq 0$. Next, take any $i, j \in \mathcal{I}$ and observe that by the concavity of the matching function,

$$\lambda m\left(o'(\delta_k, \delta_i), n'(\delta_k, \delta_i)\right) + (1 - \lambda) m\left(o''(\delta_k, \delta_i), n''(\delta_k, \delta_i)\right) \leq$$
$$m\left(\lambda o'(\delta_k, \delta_i) + (1 - \lambda) o''(\delta_k, \delta_i), \lambda n'(\delta_k, \delta_i) + (1 - \lambda) n''(\delta_k, \delta_i)\right). \tag{48}$$

Then, by continuity and strict monotonicity of $m(\cdot)$ there exists some tuple $(\tilde{o}(\delta_i, \delta_j), \tilde{n}(\delta_i, \delta_j)) \leq$

$(\lambda o'(\delta_i,\delta_j)+(1-\lambda)o''(\delta_i,\delta_j), \lambda n'(\delta_i,\delta_j)+(1-\lambda)n''(\delta_i,\delta_j))$, such that

$$m(\tilde{o}(\delta_k,\delta_i),\tilde{n}(\delta_k,\delta_i)) = \lambda m\left(o'(\delta_k,\delta_i),n'(\delta_k,\delta_i)\right)+(1-\lambda)m\left(o''(\delta_k,\delta_i),n''(\delta_k,\delta_i)\right). \tag{49}$$

Then, let $\tilde{u}$ be a tuple of market allocations such that (49) holds for all $i,j\in\mathcal{I}$. It is easy to see that $\tilde{u}\in U$ and $f(x,\tilde{u},t)=\lambda f(x,u',t)+(1-\lambda)f(x,u'',t)$. Lastly, we need to show that $\tilde{u}$ is such that $g(x,\tilde{u},t)\geq 0$. But this clearly holds since

$$\sum_j \tilde{o}(\delta_i,\delta_j) \leq \sum_j \left[\lambda o'(\delta_i,\delta_i)+(1-\lambda)o''(\delta_i,\delta_i)\right] \leq o(\delta_i), \quad \forall i \tag{50}$$

$$\sum_i \tilde{n}(\delta_i,\delta_j) \leq \sum_i \left[\lambda n'(\delta_i,\delta_i)+(1-\lambda)n''(\delta_i,\delta_i)\right] \leq n(\delta_j), \quad \forall j \tag{51}$$

Thus, $\tilde{u}\in N(x,U,t)$, which proves the claim.

Claim 3. Let $\Gamma=\{(x,u,t): g(x,u,t)\geq 0, u\in U, t\in[0,\infty)\}$ and $\Gamma_t=\{x:(x,u,t)\in\Gamma$ for some $u\in U\}$. If $x_n\in\Gamma_t$, $v_n\in N(x_n,U,t)$, $x_n\to x$, $v_n\to v$, then $x\in\Gamma_t$ and $v\in N(x,U,t)$. Observe that any tuple of state variables that satisfies the aggregate resource constraints $\sum_i o(\delta_i)=A$, $\sum_j n(\delta_j)=1-A$ is an element of Γ_t. Since all x_n satisfy those, so does x and hence $x\in\Gamma_t$. Next, let $f(x_n,u_n,t)$ be the vector induced by x_n, v_n. By continuity of the matching function, $f(x_n,u_n,t)\to f(x,u,t)$, where $f(x,u,t)$ is the vector induced by x, v. Next, it is easy to see that $f_0(x_n,t)\to f_0(x,t)$. Thus, all we are left to do to prove the claim is to show that $u\in U$ and that $g(x,u,t)\geq 0$. Since U is closed the former holds. Since $g(x_n,u_n,t)\geq 0$ for all n, then the latter holds as well. Thus, $v\in N(x,U,t)$.

Claim 4. For each $p\neq 0$ there exist locally integrable functions $\varphi_p(t)$ and $\psi_p(t)$ such that for all $(x,u,t)\in\Gamma$, $f_0(x,u,t)+p\cdot f(x,u,t)\leq\varphi_p(t)+\psi_p(t)||x||$ and there exists a constant M such that $||x(t_0)||\leq M$ for all admissible x and $f_0(x,u,t)\leq\hat{\gamma}(t)$ for all $(x,u,t)\in\Gamma$. Then, it is easy to see that $\sum_i \delta_i > exp(-rt)\sum_i o(\delta_i)\delta_i$, since $o(\delta_i)\leq A<1$. Next, observe that for $i \in [1, I]$ and $j \in [I+1, 2I]$,

$$f_i(x,u,t) \leq \gamma f(\delta_i) A + \sum_k m(o(\delta_k, \delta_i), n(\delta_k, \delta_i) \leq \gamma A + Im(1,1) \leq \gamma + Im(1,1), \tag{52}$$

$$f_j(x,u,t) \leq \gamma f(\delta_{j-I})(1-A) + \sum_k m(o(\delta_{j-I}, \delta_k), n(\delta_{j-I}, \delta_k) \leq \gamma(1-A) + Im(1,1) \leq \gamma + Im(1,1). \tag{53}$$

Then, let $\varphi_p(t) = \sum_i \delta_i + p \cdot \mathbf{M}(\mathbf{1}, \mathbf{1})$, where $\mathbf{M}(\mathbf{1}, \mathbf{1})$ is a vector of length $2I$ whose elements are $\gamma + Im(1,1)$. Then, set $\psi_p(t) = 0$ and observe that by construction $f_0(x,u,t) + p \cdot f(x,u,t) \leq \varphi_p(t) + \psi_p(t)||x||$, with both $\varphi_p(t)$ and $\psi_p(t)$ both locally integrable. Next, observe that there is a unit mass of agents in the economy and so $1 \geq ||x||$. If $\hat{\gamma}(t) = \sum_i \delta_i$, clearly $\hat{\gamma}(t) \geq f_0(x,u,t)$, for all $(x,u,t) \in \Gamma$. This proves the claim.

Claim 5. There exist integrable functions $v^i(t)$ defined on $[0, \infty)$ such that for all admissible tuples (x,u), for all t, $f_i(x,u) \leq v^i(t)$. Take $v^i(t) = \gamma + Im(1,1)$. From the preceding claim, it follows that $f_i(x,u) \leq v^i(t)$ for all t. Lastly, $v^i(t)$ is integrable, so the claim holds.

Together claims 1 through 5 establish that our problem satisfies the conditions outlined in Theorem 21 on page 406 from S&S. Thus, an optimal tuple (x,u) exists.

Characterizing Planner's Allocation. Define the Hamiltonian for the original planner's problem by

$$\begin{aligned} H \equiv & \sum_i o(\delta_i)\delta_i \\ & + \sum_i \tilde{\lambda}_i \left[\gamma f(\delta_i) \sum_k o(\delta_k) - \gamma o(\delta_i) + \sum_k m\left(o(\delta_k, \delta_i), n(\delta_k, \delta_i)\right) - \sum_j m\left(o(\delta_i, \delta_j), n(\delta_i, \delta_j)\right) \right] \\ & + \sum_j h_j \left[\gamma f(\delta_j) \sum_k n(\delta_k) - \gamma n(\delta_j) + \sum_k m\left(o(\delta_j, \delta_k), n(\delta_j, \delta_k)\right) - \sum_i m\left(o(\delta_i, \delta_j), n(\delta_i, \delta_j)\right) \right], \end{aligned} \tag{54}$$

where $\tilde{\lambda}_i$ is the co-state associated with law of motion for $o(\delta_i)$ and h_j is the co-state associated with the low of motion for $n(\delta_j)$. Thus, the Lagrangian of the problem is given by

$$\begin{aligned} L \equiv & H + \sum_i \mu_i \left(o(\delta_i) - \sum_j o(\delta_i, \delta_j) \right) + \sum_j \nu_j \left(n(\delta_j) - \sum_i n(\delta_i, \delta_j) \right) \\ & + \sum_i \sum_j \tilde{o}_{ij} o(\delta_i, \delta_j) + \sum_i \sum_j \tilde{n}_{ij} n(\delta_i, \delta_j), \end{aligned} \tag{55}$$

where μ_i, ν_j, $\tilde{o}_{ij}$, $\tilde{n}_{ij}$ are the associated Lagrange multipliers. Then, the first order conditions for optimality are given by

$$\frac{\partial L}{\partial o(\delta_i)} = r\tilde{\lambda}_i - \dot{\tilde{\lambda}}_i, \quad \forall i \tag{56}$$

$$\frac{\partial L}{\partial n(\delta_j)} = rh_j - \dot{h}_j, \quad \forall j \tag{57}$$

$$\frac{\partial L}{\partial o(\delta_i, \delta_j)} = 0, \quad \forall i, j \tag{58}$$

$$\frac{\partial L}{\partial n(\delta_i, \delta_j)} = 0, \quad \forall i, j \tag{59}$$

$$\mu_i \left(o(\delta_i) - \sum_j o(\delta_i, \delta_j) \right) = 0, \quad \forall i \tag{60}$$

$$\nu_j \left(n(\delta_j) - \sum_i n(\delta_i, \delta_j) \right) = 0, \quad \forall j \tag{61}$$

$$\tilde{o}_{ij} o(\delta_i, \delta_j) = 0, \quad \forall i, j \tag{62}$$

$$\tilde{n}_{ij} n(\delta_i, \delta_j) = 0, \quad \forall i, j \tag{63}$$

$$\lim_{t \to \infty} exp(-rt)\tilde{\lambda}_i o(\delta_i) = 0, \quad \forall i \tag{64}$$

$$\lim_{t \to \infty} exp(-rt) h_j n(\delta_j) = 0, \quad \forall j \tag{65}$$

We can reduce the first four first order conditions to the following system

$$(r+\gamma)\tilde{\lambda}_i = \delta_i + \gamma \mathbb{E}\tilde{\lambda} + \mu_i + \dot{\tilde{\lambda}}, \tag{66}$$

$$(r+\gamma)h_j = \gamma \mathbb{E}h + \nu_j + \dot{h}, \tag{67}$$

$$\mu_i = \lambda(q(\delta_i,\delta_j))\eta(q(\delta_i,\delta_j))\left[\tilde{\lambda}_j - h_j - (\tilde{\lambda}_i - h_i)\right] + \tilde{o}(\delta_i,\delta_j), \tag{68}$$

$$\nu_j = \frac{\lambda(q(\delta_i,\delta_j))}{q(\delta_i,\delta_j)}\left[1-\eta(q(\delta_i,\delta_j))\right]\left[\tilde{\lambda}_j - h_j - (\tilde{\lambda}_i - h_i)\right] + \tilde{n}(\delta_i,\delta_j), \tag{69}$$

where $\mathbb{E}\tilde{\lambda}$ and $\mathbb{E}h$ are the average $\tilde{\lambda}_i$ and h_j. Then, noting that in steady state the values of $\tilde{\lambda}_i$ and h_j are constant, it follows that we can reduce the system above to

$$(r+\gamma)\tilde{\lambda}_i = \delta_i + \gamma \mathbb{E}\tilde{\lambda} + \lambda(q(\delta_i,\delta_j))\eta(q(\delta_i,\delta_j))\left[\tilde{\lambda}_j - h_j - (\tilde{\lambda}_i - h_i)\right] + \tilde{o}(\delta_i,\delta_j), \tag{70}$$

$$(r+\gamma)h_j = \gamma \mathbb{E}h + \frac{\lambda(q(\delta_i,\delta_j))}{q(\delta_i,\delta_j)}\left[1-\eta(q(\delta_i,\delta_j))\right]\left[\tilde{\lambda}_j - h_j - (\tilde{\lambda}_i - h_i)\right] + \tilde{n}(\delta_i,\delta_j). \tag{71}$$

With appropriate relabeling, these conditions are exactly the same as the Bellman equations along the decentralized equilibrium, (10) and (11), when $\tilde{o}(\delta_i,\delta_j) = \tilde{n}(\delta_i,\delta_j) = 0$. But, the multipliers are zero if and only if the planner assigns positive measures of owners and non-owners on this market. That is, the planner's conditions coincide with the decentralized ones along the equilibrium path of trade only when the planner assigns positive measures on that market. If there are no owners assigned on that market, this means that $\tilde{o}(\delta_i,\delta_j) > 0$ and so the value of an owner participating in that market is not large enough to yield her optimal value, i.e. the owner can do better (is more socially beneficial) if she is on a different market. Analogously for non-owners: if $\tilde{n}(\delta_i,\delta_j) > 0$ there are no non-owners on that market and the socially optimal value h_j is higher than any potential payoff that the non-owner might earn from participating that market, i.e. the non-owner cannot be compensated with her market utility from participating in the market. Thus, the decentralized equilibrium is efficient.

Sufficiency of the First Order Conditions. Next, observe that the maximand as well as constraints are continuously differentiable. Let $A(t)$ be the set of all tuples of owners, $o(\delta_i, \delta_j)$ and non-owners, $n(\delta_i, \delta_j)$, assigned to each market such that constraints (25), (26), (27), (28) are satisfied given the masses of owners, $o(\delta_i)$, and non-owners, $n(\delta_j)$, at time t. Thus, it is easy to see that this set is convex for all t. Let $\hat{x}(t)$ be a tuple of owner masses, $o(\delta_i)$, non-owner masses, $n(\delta_j)$, and owner and non-owner market participation assignments, $o(\delta_i, \delta_j)$, $n(\delta_i, \delta_j)$ such that the constraints of the problem and the first order conditions are satisfied. Given the resulting co-states $\hat{\lambda}_i$ and h_j, define $\hat{H}$ to be the maximized Hamiltonian, i.e. $\hat{H} \equiv \max_{o(\delta_i,\delta_j),o(\delta_i,\delta_j)\in A(t)} H$. Since any tuple of owner and non-owner market assignments which is an element of $\arg\max_{o(\delta_i,\delta_j),o(\delta_i,\delta_j)\in A(t)} H$ is independent of the state variables $o(\delta_i)$, $n(\delta_j)$, it is easy to see that the resulting maximized Hamiltonian is linear in the state variables and as a consequence jointly concave in them. Hence, by Arrow's sufficiency theorem $\hat{x}(t)$ is a global maximum.

$\blacksquare$