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Asset Liquidity and Indivisibility

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Abstract

We study asset liquidity in a search-theoretic framework where divisible assets can facilitate exchange for an indivisible consumption good. The distinctive characteristics of our theory are that the asset dividend can be either positive or negative and buyers can choose whether or not to carry the asset and trade for the indivisible good. Buyers' participation determines the demand for asset liquidity and hence asset price carries a component of liquidity premium to reflect its function of trade facilitation. The economy features multiple equilibria when the asset dividend is negative, due to the trade-off between the probability of trade and the endogenous cost of holding the asset.

JEL: D51, E40, G12.

Keywords: Asset, Indivisibility, Liquidity, Search.

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1 Introduction

Assets, just like money, can convey liquidity for consumption purposes. Investors value assets for their fundamental value measured in dividend yields as well as their liquidity. One example of such assets is the US Treasury bill. As Krishnamurthy and Vissing-Jorgensen (2012) point out, asset holders are willing to pay a premium of no more than 46 base points for the liquidity attribute of US Treasuries. In a growing literature based on the New Monetarist framework of Lagos and Wright (2005), this liquidity function of assets has been modeled as the essential medium of exchange in a market with trading frictions.

We examine an economy in which buyers can hold divisible assets and use them to buy one unit of indivisible consumption good from sellers in a frictional market. We focus on asset liquidity and our analysis differs from the previous studies in three aspects. First, the good that buyers want to trade assets for is indivisible. The divisibility of goods (or discrete multi-units) is a convenient abstraction at the aggregate level. However, at the level of pairwise trades, a typical buyer consumes a limited or a unique amount of many goods. In particular, buyers are likely to liquidate assets to consume big-ticket items rather than everyday items.[1] The indivisible nature of the good has important implications on equilibrium by affecting the trading price.

Second, with indivisible goods, agents lose an intensive margin of adjustment during trade, and hence we allow free entry by buyers, i.e., adjustment on the extensive margin. Since buyers of the indivisible goods are also asset holders, their participation in the frictional market endogenously generates demand for liquidity, and thus we establish a link between entry and asset price.

Third, the asset that provides liquidity is assumed to be in fixed total supply and it bears a deterministic exogenous dividend. While the previous study exclusively focuses on positive dividend, we also consider the case that the dividend is negative. Apart from being merely an exercise for theoretical completeness, we think the negative dividend is real, such as the storage cost of holding gold, an asset and commodity money. While positive dividend generates a yield, negative dividend puts a cost on asset holders who

[1] Big ticket items also tend to be durable goods. As an abstraction, we model the indivisible goods as being perishable. However, in a stationary environment, perishability or durability of the indivisible goods will not change the asset pricing results.

care about its liquidity, hence affecting asset price and equilibrium outcome.

We assume agents use either generalized Nash bargaining or price posting with competitive search to determine the price of the indivisible goods. The former is an ex-post mechanism mapping the amount of asset holding into outcomes in bilateral trade, while the latter is an ex-ante mechanism mapping the posted terms of trade into the choice of asset holding. Under bargaining, buyers commit to bringing the lowest amount of assets needed to make sellers indifferent between trading or not, and the bargained price does not depend on the bargaining power or the asset dividend. Price posting with competitive search entails commitment by sellers prior to the trade, and buyers choose their asset holdings after observing prices.

We show that, under both trading mechanisms, there is a unique equilibrium when the asset dividend is positive, and in general, multiple equilibria exist for a negative dividend. Multiplicity is due to the trade-off between the probability of trade and the opportunity cost of holding assets. When many buyers enter the frictional market, high participation lowers the probability of trade but also drives up the liquidity demand for assets. Hence, asset price increases and the cost of holding assets decreases, compensating buyers for the lower trading probability. The key to this result is the interaction between free entry and endogenous asset pricing with a negative dividend. For a positive dividend, the relationship between asset price and the cost of holding assets reverses. The above trade-off disappears and the equilibrium is unique.

An important implication of indivisibility is that some buyers may choose not to participate in the frictional market in equilibrium. This is in contrast to models with divisible goods where buyers always have full entry, as in Lagos and Rocheteau (2007). With indivisible goods, only the price can adjust but not quantity, and hence the economy needs a lot of liquidity to support all the buyers trading in the frictional market. This only happens when the asset dividend is high enough. If the total liquidity provided by assets is so abundant, all buyers participate and the asset is priced at the fundamental value. If the asset liquidity is less abundant caused by a smaller dividend, some buyers may start to drop out of the frictional market. When liquidity is scarce, assets are priced above the fundamental value to satisfy the liquidity demand, and there is a liquidity premium in the asset price to reflect its function in facilitating transactions. Eventually, the market will shut down if the asset dividend is too low to support the frictional trade.

As is standard in non-convex economies, we also introduce lotteries. While lotteries serve as a threat of not delivering the goods and help sellers to extract some trading surplus under bargaining, they make no impact under competitive search. We show that lotteries are not used in equilibrium since it is mutually beneficial to increase the expected surplus from trade by raising the trading probability.

1.1 Literature

The New Monetarist environment, based on the seminal work of Lagos and Wright (2005), has demonstrated that, because of the liquidity function of assets, asset prices can carry a liquidity premium and move away from fundamental prices. Hence, assets are valued for both their dividend yield and liquidity service. The case of divisible goods traded for assets has been studied extensively in, e.g., Wallace (2000), Geromichalos et al. (2007), and Rocheteau and Wright (2013), but the equilibrium consequences of indivisible goods traded for divisible assets have been neglected in the literature. One notable exception is Rabinovich (2017), who studies the trade of indivisible goods with divisible assets under price posting with random search. He finds a unique equilibrium when assets are priced at the fundamental value, and multiple equilibria exist when assets are valued for their liquidity component. Han et al. (2016) study indivisible goods traded with divisible means of payment in the form of money or credit. As in the current paper, they analyze equilibrium under bargaining and competitive search. Wang et al. (2019) study indivisible goods traded with money and credit, and they consider price posting with random search. However, the cost of holding money or using credit is exogenous, whereas the cost of holding assets is endogenous.

Indivisibility of goods instead of assets matters and the consequences for equilibria differ from models with indivisible assets/money as in Shi (1995), Trejos and Wright (1995), and Wallace (2000).[2] Using the Shi-Trejos-Wright framework, Wallace (2000) considers indivisible assets traded for divisible goods with heterogeneity in positive asset dividends to obtain an equilibrium liquidity structure of asset yields. Duffie et al. (2005) study Over-The-Counter (OTC) markets in a simplified Shi-Trejos-Wright environment

[2] In Shi-Trejos-Wright (1995) and Wallace (2000), agents cannot accumulate more than one unit of money. See Julien et al. (2016), He and Wright (2019), and Wright et al. (2019) for recent models based on Shi-Trejos-Wright using assets instead of fiat money.

with linear utility.

Starting from Geromichalos et al. (2007), there is a literature studying assets in fixed supply as a medium of exchange for consumption in the New Monetarist environment. In a model with divisible goods and assets, they allow fiat money and real financial assets to compete as the media of exchange and focus on the equilibrium link between monetary policy and asset price. Lagos and Rocheteau (2009) also study OTC markets in a New Monetarist environment with heterogeneous demands for divisible assets. They show that individual responses of asset demand constitute a fundamental feature of illiquid markets and are key determinants of trade volume, bid-ask spreads, and trading delays. These papers focus on the trading frictions in OTC asset markets and the effect of frictions on asset prices.

Lagos (2011) considers an exchange economy where assets and money are used as the medium of exchange for a divisible good. Like us, assets are traded in a frictionless and competitive market, and he studies the determinant of asset prices when assets are the only medium of exchange. Similarly, Rocheteau and Wright (2013) also consider the case with the asset being the only medium of exchange. They find multiple stationary equilibria, across which asset prices, market participation, output, and welfare are positively correlated, and they generate a variety of nonstationary equilibria. While we endogenize the participation of buyers via a market utility condition, they endogenize participation by free entry of sellers in the frictional market, and hence they do not have a connection between asset demand and entry. Both Lagos (2011) and Rocheteau and Wright (2013) only study positive dividend values while we allow the dividend to be positive or negative.[3]

In the above papers, as in ours, asset prices carry a liquidity premium because they serve directly as a medium of exchange. A highly related literature, starting from Geromichalos and Herrenbrueck (2016) and followed by Geromichalos and Herrenbrueck (2017) and Geromichalos and Jung (2018), define and discuss indirect asset liquidity. In those papers, assets are viewed as a substitute for money, and asset prices carry a liquidity premium because agents sell assets for cash in secondary OTC markets. Hence, assets can provide extra liquidity when money is not enough, and the liquidity function of assets is indirect. Geromichalos and Herrenbrueck (2018) study the joint determination of asset supply and liquidity in a model where financial assets provide liquidity indirectly. A key finding is

[3] Lagos (2011) further assumes that the dividend value is determined stochastically.

that an asset's liquidity depends on the exogenous characteristics of the OTC markets and the endogenous trading behavior of agents in those markets. While we focus on direct asset liquidity in the current paper, we also show that exogenous asset dividend and endogenous market participation jointly affect asset prices.

Finally, a version of positive or negative dividend is studied in Nosal et al. (2019), where they focus on the endogenous formation of middlemen. They assume that holding inventories of assets entails a positive return, as the positive dividend in our paper, and holding goods entails a cost, i.e., a negative dividend. They find that multiple equilibria can only arise with a positive dividend and endogenous market composition. The source of multiplicity is the interaction between the middleman's inventory decision and their probability of trade with producers. While we also find equilibrium multiplicity, it happens with a negative dividend and endogenous participation. In our paper, multiplicity arises due to the trade-off between the probability of trade and the opportunity cost of carrying assets, which depends on the size of participating agents. Such an interaction between the extensive margin and the carrying cost does not exist in Nosal et al. (2019).

The rest of the paper is organized as follows. Section 2 and 3 describe the environment and stationary equilibrium. In Section 4, we introduce lotteries. Section 5 concludes.

2 The Environment

The environment is based on Rocheteau and Wright (2005).[4] Time is discrete and a continuum of buyers and sellers, with measures N and 1, live forever. Each period, $n \leq N$ buyers and all sellers participate in two consecutive markets. The first market is a frictional decentralized market (DM). In the DM, meetings occur according to a general meeting technology, which is homogeneous of degree one. Given the buyer-seller ratio n, which is also the measure of participating buyers in the DM, the meeting probabilities for sellers and buyers are $\alpha(n)$ and $\alpha(n)/n$, respectively. Assume $\alpha' > 0$, $\alpha'' < 0$, $\alpha(0) = 0$, $\lim_{n\to\infty} \alpha(n) = 1$, and $\lim_{n\to 0} \alpha'(n) = 1$. The second market is a frictionless centralized market (CM). Agents discount between periods with $\beta \in (0, 1)$, but not across markets

[4] The original alternating markets framework by Lagos and Wright (2005) has agents receiving a preference shock in the CM, revealing whether they will be a buyer or a seller in the DM. In Rocheteau and Wright (2005), buyers are always buyers and sellers are always sellers. All our results hold for both frameworks.

within a period, and $r = 1/\beta - 1$ is the discount rate.

Both buyers and sellers consume a divisible good in the CM, while only buyers consume and sellers produce an indivisible good in the DM. Buyers' preferences are given by $U(x) - h + u\mathbf{1}$, where x is CM consumption, h is CM labor, u is DM utility from the indivisible good, and $\mathbf{1}$ is an indicator function, yielding 1 if trade occurs and 0 otherwise. Let x be the numeraire, and we assume that x is produced one-to-one from h. Sellers' preferences are $U(x) - h - c\mathbf{1}$ and $c < u$ is the cost of producing the DM good.

The only asset in the economy is a real asset, a, which is perfectly divisible and recognizable. It cannot be counterfeited. The total asset supply is fixed at A^s and at $t = 0$ each buyer is endowed with A^s/N assets. In subsequent periods the asset is traded competitively in the CM at price φ. The real asset generates an exogenously determined dividend ρ, paid in terms of x in the CM. The dividend ρ can be either positive or negative. If $\rho < 0$, it is a storage cost of holding the asset, as in Kiyotaki and Wright (1989).

Because of the trading frictions in the DM, agents cannot commit and there are no enforcement or punishment mechanisms. Hence, buyers must bring a medium of exchange into the DM to pay sellers and they use the asset for payments.[5] The DM trade implies a price and quantity bundle $(p, q) \in \mathbf{P} \times \mathbf{Q}$, where $\mathbf{P} = \{0 \leq p \leq L\}$ and $\mathbf{Q} = \{0, 1\}$. We use L to denote the total liquidity in the economy, with $L = (\varphi + \rho)a$ being the value of a buyer's asset holding in the DM. After a successful trade, sellers acquire the assets and then use them in the subsequent CM.

3 The Model

Let $W_t(a)$ and $V_t(a)$ denote the value functions of an agent holding a units of assets when entering the CM and the DM, respectively. Buyers in the CM obtain

$$W_t^b(a) = \max_{x,h,\hat{a}} \left\{ U(x) - h + \beta V_{t+1}^b(\hat{a}) \right\} \text{ st. } x = (\varphi_t + \rho)\, a + h - \varphi_t \hat{a},$$

[5] Alternatively, one can assume that agents use assets as collateral to get credit in the DM, as in Kiyotaki and Moore (1997) with the pledgeability parameter to be one. All the results still hold. Lagos et al. (2017) have shown that these two setups are mathematically equivalent.

where $\hat{a}$ is the asset holding carried into the following DM. Buyers participate in the DM if $V_{t+1}^{b} \geq 0$. Eliminating h from the budget constraint and solving for optimal x^{*} yields,

$$W_t^b(a) = \Sigma + (\varphi_t + \rho)\, a + \max_{\hat{a}} \left\{ \beta V_{t+1}^b(\hat{a}) - \varphi_t \hat{a} \right\}, \tag{1}$$

where $\Sigma = U(x^{*}) - x^{*}$ and $U'(x^{*}) = 1$. Similarly, for a seller with a we have

$$W_t^s(a) = \Sigma + (\varphi_t + \rho)\, a + \max_{\hat{a}} \left\{ \beta V_{t+1}^s(\hat{a}) - \varphi_t \hat{a} \right\}. \tag{2}$$

The buyer's payoff in the DM is

$$V_t^b(a) = \frac{\alpha(n)}{n} \left[u + W_t^b\left(a - \frac{p}{\varphi_t + \rho} \right) \right] + \left[1 - \frac{\alpha(n)}{n} \right] W_t^b(a), \tag{3}$$

where p is the price of the DM good measured by the numeraire. Using $\partial W_t^b / \partial a = \varphi_t + \rho$, we can simplify (3) as

$$V_t^b(a) = \frac{\alpha(n)}{n}(u - p) + W_t^b(a). \tag{4}$$

Similarly, for sellers, we have

$$V_t^s(a) = \alpha(n)(p - c) + W_t^s(a). \tag{5}$$

Since sellers do not need to pay in the DM, a necessary condition for them to hold assets in the DM is $\varphi_t \leq \beta(\varphi_{t+1} + \rho)$. In a steady state equilibrium, it implies $\varphi \leq \rho / r \equiv \varphi^F$, and φ^F is the fundamental price of the asset. Hence, sellers only hold assets as a store of value when they are priced at the fundamental value.

In the next two sections, we analyze the DM trade of an indivisible good and assets under generalized Nash bargaining and competitive search, and study its implications on asset pricing and the good's price in the DM.

3.1 Bargaining

First, we consider the case that the price of the indivisible good is determined by generalized Nash bargaining. In the DM, buyers and sellers face the following problem,

$$\max_{p} (u - p)^{\eta} (p - c)^{1-\eta} \text{ st. } p \leq (\varphi_t + \rho)\, a,\ u - p \geq 0,\ p - c \geq 0. \tag{6}$$

When the asset is costly to carry, buyers have no incentives to bring more assets than p. The feasibility constraint $p = (\varphi_t + \rho)a$ is binding and hence $c \leq (\varphi_t + \rho)a \leq \bar{p}^b$, where $\bar{p}^b = (1-\eta)u + \eta c$ is the unconstrained bargaining solution. Any negotiated price $p \in [c, \bar{p}^b]$ is a potential bargaining solution.[6] Substituting V_{t+1}^b into W_t^b and a buyer's CM value function is

$$W_t^b(a) = \Sigma + (\varphi_t + \rho)\,a + \beta W_{t+1}^b(0) + \max_{\hat{a}} \left\{ \beta \frac{\alpha(n)}{n}(u-p) + \left[\beta\left(\varphi_{t+1} + \rho\right) - \varphi_t\right]\hat{a} \right\}.$$

To ease the presentation of the buyer's problem, we follow Rocheteau and Rodriguez-Lopez (2014) to define $s_t = \varphi_t/\beta(\varphi_{t+1} + \rho) - 1$ as the spread of the asset,[7] which can be viewed as the liquidity premium of holding the asset. Then, the buyer's problem can be rewritten as

$$\tilde{V}^b(n, s_t) \equiv \max_{\hat{a} \in [\underline{a}, \bar{a}]} \beta \left\{ \frac{\alpha(n)}{n}\left[u - \left(\varphi_{t+1} + \rho\right)\hat{a}\right] - s_t\left(\varphi_{t+1} + \rho\right)\hat{a} \right\}, \tag{7}$$

where $\underline{a} = c/(\varphi_{t+1} + \rho)$ and $\bar{a} = [(1-\eta)u + \eta c]/(\varphi_{t+1} + \rho)$. It is apparent that $\tilde{V}^b$ is strictly decreasing in $\hat{a}$ for all values of $s_t > 0$. The optimal solution satisfies $\hat{a}(\varphi_{t+1} + \rho) = c$. With bargaining, a buyer can commit to not paying more than the seller's reservation price c, as they have the first-mover advantage. The buyer's surplus from trade decreases if they carry an amount larger than c, and this is true even in the case of $s_t = 0$, when all the buyers' liquidity need is satisfied and sellers hold the rest of the assets. Hence, the bargaining outcome $p = c$ is independent of the cost of holding assets. Notice that the same result holds in a monetary environment in Han et al. (2016). They demonstrate that when buyers bring real monetary balances into the DM, the bargained price reduces to $p = c$ even at $i = 0$. Therefore, under bargaining and indivisible goods, when a buyer brings a medium of exchange into the DM, he is able to extract all the surplus from trade.

[6] Using proportional bargaining, i.e., Kalai and Smorodinsky (1975), for terms of trade yields the same results.

[7] Using $\beta = 1/(1+r)$, rewrite the spread equation as $1 + s_t = (1+r)\varphi_t/(\varphi_{t+1} + \rho)$. This is reminescent of the Fisher equation used in monetary models, where $1 + i = (1+r)\phi_t/\phi_{t+1}$, with i being the nominal interest rate set by monetary policy and ϕ the price of money in terms of the numeraire goods in the CM. Thus, i is the spread on money. With money, i is exogenous, while with assets, the spread s is endogenous.

The buyer's expected value from participating in the DM can now be rewritten as

$$\tilde{V}^b(n_t, s_t) = \beta\left[\frac{\alpha(n_t)}{n_t}(u-c) - s_t c\right] \geq 0, \tag{8}$$

which is the discounted expected total benefit net of the cost of carrying the asset. The measure of DM buyers $n_t \leq N$ is determined by a free entry condition, $\tilde{V}^b(n_t, s_t) \geq 0$. If $s_t = 0$, the asset is priced at its fundamental value and $\tilde{V}^b(n_t, s_t) > 0$, $\forall n_t \leq N$. Hence, all the buyers participate in the DM. If $s_t > 0$, buyers need to pay a cost of holding assets to enter the DM. When the cost is large enough, buyers start to drop out of the DM. Hence, $\tilde{V}^b(n_t, s_t) = 0$ if $n_t < N$ and $\tilde{V}^b(n_t, s_t) > 0$ if $n_t = N$.

We focus our attention on a stationary equilibrium with $\varphi_t = \varphi$ and $s_t = s$, $\forall t$. To establish equilibrium existence and uniqueness, we start with defining the aggregate demand, $L^d(s)$, and the aggregate supply, $L^s(s)$, of liquidity. Under bargaining, $L^d(s) = n(s)c$ and $L^s(s) = (\varphi(s) + \rho)A^s$, where $n(s)$ is determined from the free entry condition (8) and $\varphi(s)$ is solved from the definition of spread,

$$\varphi(s) = \frac{\beta\rho(1+s)}{1-\beta(1+s)} = \frac{\rho(1+s)}{r-s}. \tag{9}$$

When $s = 0$, we have $\varphi = \rho/r \equiv \varphi^F$. The asset is priced fundamentally and does not carry a liquidity premium. In this case, there is enough liquidity for transactional needs, i.e., $L^s(s) > L^d(s)$, and the marginal holder of assets is a seller. For $s > 0$ and $s \neq r$, we have

$$L^d(s) = n(s)c = \frac{\rho(1+r)}{r-s}A^s = L^s(s), \tag{10}$$

and the supply of liquidity is infinitely elastic in the special case of $s = r$. From (10), we can solve for

$$s = r - \frac{\rho A^s(1+r)}{nc}, \tag{11}$$

and n is determined by (8). Substituting (11) into (9) yields the asset price

$$\varphi^n = \frac{nc}{A^s} - \rho. \tag{12}$$

Evaluating (12) at $n = N$, we have

$$\varphi^N = \frac{Nc}{A^s} - \rho > \varphi^F, \tag{13}$$

and the asset price with full entry is strictly decreasing in ρ and A^s.

Definition 1 *A stationary bargaining equilibrium (SBE) is a list $\{p^b, s, n\}$ such that: (i) p^b solves (6); (ii) $s > 0$ solves $L^s = L^d$ or $s = 0$; (iii) $n \leq N$ solves (8).*

To fully describe the equilibrium, we start with two lemmas characterizing the demand and supply of liquidity. First define s^N as the cutoff spread value at which all the buyers are willing to enter the DM. Let $\bar{s}$ be the highest value of spread that gives buyers zero surplus from the DM trade. Furthermore, let ρ^F, solving (10) with $n = N$, be the cutoff dividend value above which assets are priced fundamentally, corresponding to $s = 0$, i.e., no liquidity premium.

Lemma 1 *There exist $\bar{s} \geq r$ and $s^N \leq \bar{s}$, such that: (i) for $s \leq s^N$, $\exists!$ L^d with $n = N$ and $L^d = Nc$; (ii) for $s \in (s^N, \bar{s}]$, $\exists!$ L^d with $n < N$, $L^d = nc$ and $dL^d/ds < 0$; (iii) for $s > \bar{s}$, $\nexists$ $n > 0$ and L^d is not well-defined.*

Lemma 2 *There exists $\rho^F > 0$, such that: (i) For $\rho < 0$ $(r < s)$, L^s is convex and $dL^s/ds < 0$; (ii) for $\rho = 0$, L^s is perfectly elastic at $s = r$; (iii) for $\rho \in (0, \rho^F)$ $(0 < s < r)$, L^s is convex and $dL^s/ds > 0$; (iv) for $\rho \geq \rho^F$, L^s is perfectly elastic at $s = 0$.*

The two lemmas are depicted visually in Figure 1. In equilibrium, a positive dividend value corresponds to a unique spread, while for a negative ρ, there may be multiple s. Since the asset price is a function of s, as shown in (9), this leads to multiple equilibria for $\rho < 0$, as summarized in Proposition 1.

Proposition 1 *There exists $\underline{\rho} < 0$, such that: (i) for $\rho \geq 0$ or $\rho = \underline{\rho} < 0$, $\exists!$ SBE; (ii) for $\rho \in (\underline{\rho}, 0)$, $\exists$ two SBE.*

Note that, we can rewrite the participation constraint from (8) and (11) as

$$\begin{aligned} \tilde{V}^b(n) &= \beta\frac{\alpha(n)}{n}(u - c) - (1-\beta)c + \frac{\rho A^s}{n} \\ &= B(n) + \frac{\rho A^s}{n} \geq 0, \end{aligned}$$

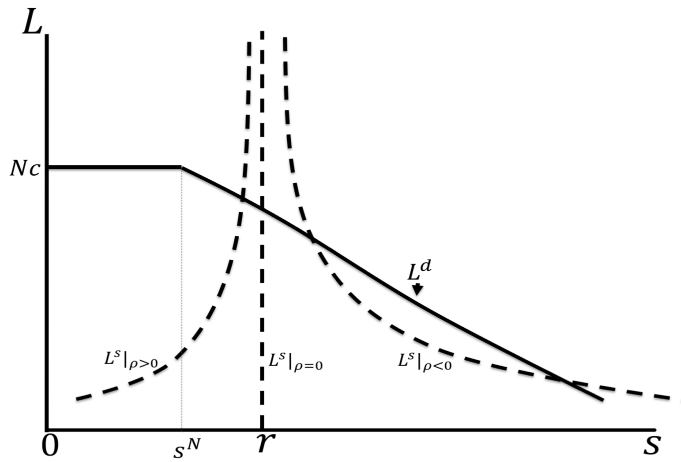

Figure 1: Supply and Demand of Liquidity with Bargaining

where $B(n)$ is the buyer's gain from DM trade. Evaluating $\tilde{V}^b(n)$ over $[0, N]$ implies the existence of a unique $\underline{\rho}$, such that for $\rho \geq \underline{\rho}$, the buyers who enter the DM can get a positive surplus. Interestingly, the equilibrium level of participation corresponding to $\underline{\rho}$ is characterized by

$$\alpha'(n) = \frac{(1-\beta)c}{\beta(u-c)},$$

which is unique and relates to the well-known Hosios (1990) condition for efficient entry.[8] The marginal contribution to the matching process by a buyer equals the flow cost of entry, or equivalently, the buyer's share over the discounted total surplus. Therefore, the endogenous entry of buyers is constrained efficient at $\underline{\rho}$.

Proposition 1 shows that, there exists a unique stationary equilibrium for all positive dividend values, while for negative ρ, there are exactly two equilibria. To understand the result, note that from (9), the asset price φ is negatively related to s when $\rho < 0$. When the equilibrium participation is high (low), a large (small) liquidity demand drives up (down) the asset price, implying a small (large) spread, making entry less (more) costly. On the other hand, when the entry level is high (low), buyers face a low (high) probability of trade, due to the congestion effect in the matching process. This is easily represented by

$$\tilde{V}^b(n_\ell) = B(n_\ell) + \frac{\rho A^s}{n_\ell} = B(n_h) + \frac{\rho A^s}{n_h} = \tilde{V}^b(n_h),$$

[8]The Hosios' entry condition is expressed as $\varepsilon(n) = (1-\beta)c/[\beta(u-c)\alpha(n)/n]$ with $\varepsilon(n) = \alpha'(n)n/\alpha(n)$.

with $B(n_\ell) > B(n_h)$ and $\rho A^s/n_\ell < \rho A^s/n_h$ since $\rho < 0$. Hence, buyers face a trade-off between the cost of carrying liquidity and the probability of trade. Note that when $\rho \geq 0$, $\partial\varphi/\partial s \geq 0$, and such a trade-off disappears. More (less) participation implies a smaller (larger) probability of trade and a larger (smaller) spread, and hence the equilibrium is unique.

A similar trade-off, between the probability of trade and the trading quantity in the DM, generates multiple equilibria in Rocheteau and Wright (2005). In their monetary model, there is free entry by sellers and the cost of holding liquidity is exogenously given at i, the nominal interest rate, while we consider free entry by buyers and s is endogenously determined by the demand and supply of assets. In their economy, as the level of participation and the probability of trade change in the DM, buyers bargain with sellers to settle on a different trading quantity, hence making sellers indifferent.

Define ρ^N by $\tilde{V}^b(N) = 0$, and it is the cutoff value at which every buyer is willing to enter the DM. When N is large, in order to have all the buyers participate, we need a lot of liquidity and hence $\rho^N > 0$ needs to be large. When N is small, the total liquidity needed to support a full-participation equilibrium is also small, and we can have negative dividend values $\rho^N < 0$.

These cutoffs, ρ^N, ρ^F, and $\underline{\rho}$ together characterize intervals of ρ in which: (i) liquidity provided by assets is so abundant that assets are priced fundamentally with all buyers participating in the DM, (ii) assets are still abundant to support full entry by buyers and priced above the fundamental value, (iii) less abundant to support partial participation, or (iv) so scarce that no buyers enter the DM. We now discuss these different cases and examine the equilibrium effect of changing the dividend value.

Figure 2 illustrates how the equilibrium asset price, entry, and spread change w.r.t. ρ when $\rho^N > 0$, and the case of $\rho^N < 0$ is shown in Figure 3. We mostly focus on $\rho^N > 0$ in the following discussion and the same logic applies to $\rho^N < 0$. When $\rho > \rho^F$, liquidity is abundant in the economy. Assets are priced at the fundamental value and have no liquidity premium, i.e., $s = 0$. Since carrying assets is costless, the marginal asset holder is a seller. All buyers participate in the DM due to zero holdup cost.

For $\rho \in (\rho^N, \rho^F)$, liquidity is relatively scarce but still enough to support full entry by buyers. The buyer's liquidity demand drives up the asset price to be above its fundamental value, and assets become too expensive for sellers to hold since $s > 0$. Buyers pay a

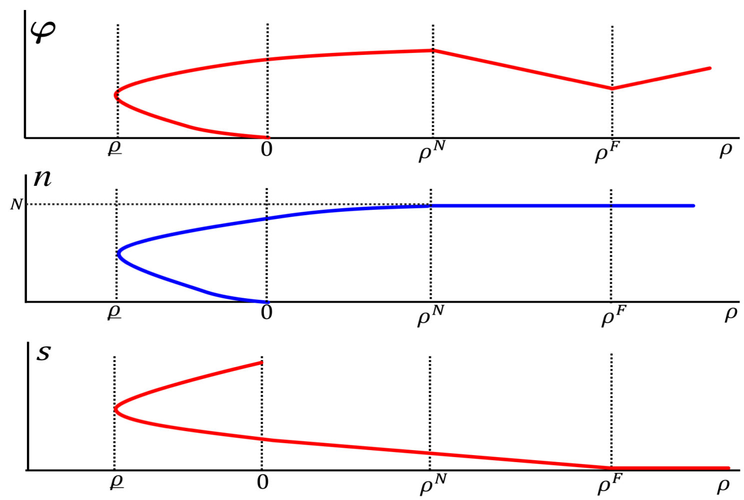

Figure 2: Bargaining Equilibrium, $\rho^N > 0$

liquidity premium for carrying assets so that they can trade in the DM. When ρ decreases, while all buyers still participate, the asset price increases to meet the liquidity demand, implying a larger liquidity premium. This result echoes a key point in the New Monetarist literature: liquidity plays a key role in determining the price of an asset.

As ρ decreases further and $\rho < \rho^F$, liquidity becomes even more scarce, and buyers start to drop out of the DM. The congestion effect is now too strong to generate a positive surplus for full entry. For $\rho < 0$, the adjustment on the extensive margin leads to two opposite effects. As ρ decreases, the asset price drops and more buyers choose not to enter the DM, while the participating buyers enjoy a higher probability of trade. This is the standard "hot potato" effect: people trade faster as the cost of transaction increases. On the other hand, there is an incentive for more buyers to participate, which leads to a higher demand for assets in DM transactions. Consequently, a larger demand will drive up the asset price, increase the liquidity premium to offset the negative dividend, and lower the cost of carrying liquidity. This channel works as if more buyers get involved to share the unpleasant nature of the assets, i.e., a negative dividend, and we name it the "stinky fish" effect following Kiyotaki and Wright (1989). The smell of the fish is not desirable but it can provide the liquidity that buyers need in the DM.

Proposition 1 shows that two equilibria exist for $\rho < 0$, with two levels of participation $n_\ell < n_h$ and two different asset prices $\varphi^{n_\ell} < \varphi^{n_h}$. In the equilibrium with a high entry n_h, the "hot potato" effect dominates, and both entry and asset price decrease as the dividend

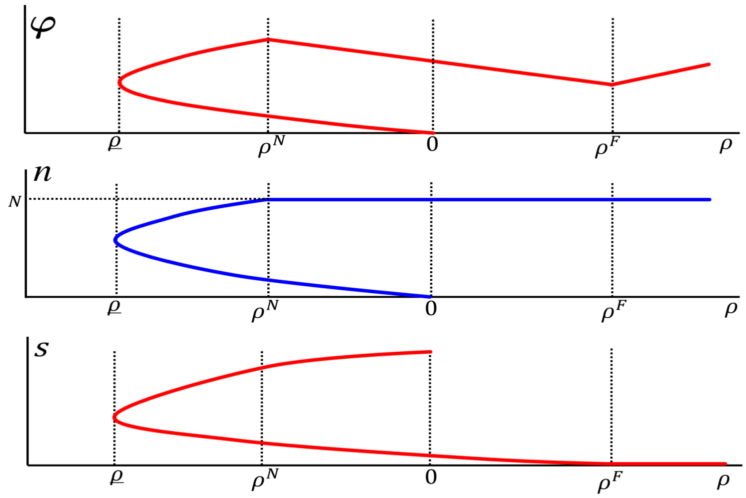

Figure 3: Bargaining Equilibrium, $\rho^N < 0$

decreases. In the equilibrium with n_ℓ, the "stinky fish" effect prevails, and n_ℓ and φ^{n_ℓ} increase as ρ decreases. If $\rho^N < 0$, as shown in Figure 3, the total measure of buyers is small and some negative dividend values can still support full entry, hence $n_h = N$ for $\rho \in [\rho^N, 0]$.

Note that we need both adjustable extensive margin and endogenous liquidity cost to have the "stinky fish" effect. Lagos and Rocheteau (2005) generates the "hot potato" effect but not the second one, since agents do not have a participation decision. Liu et al. (2011) study the "hot potato" effect through the extensive margin, but the cost of carrying liquidity is exogenous in their model. In fact, the "stinky fish" effect does not exist in any monetary models, because the cost of holding money is always exogenous and independent of participation.

Finally, Proposition 2 summarizes the effects of changing ρ in equilibria.

Proposition 2 *In the unique SBE ($\rho > 0$) and the SBE with high participation ($\rho < 0$): (i) for $\rho \geq \rho^F$, we have $\partial\varphi/\partial\rho > 0$, $n = N$ and $s = 0$; (ii) for $\rho \in [\rho^N, \rho^F)$, $\partial\varphi/\partial\rho < 0$, $n = N$ and $\partial s/\partial\rho < 0$; (iii) for $\rho \in (\underline{\rho}, \rho^N)$, $\partial\varphi/\partial\rho > 0$, $\partial n/\partial\rho > 0$, and $\partial s/\partial\rho < 0$. In the SBE with low participation ($\rho < 0$), for $\rho \in (\underline{\rho}, 0)$, $\partial\varphi/\partial\rho < 0$, $\partial n/\partial\rho < 0$, and $\partial s/\partial\rho > 0$.*

3.2 Competitive Search

In this section, we study competitive search equilibrium with price posting.[9] Similar to Moen (1997) and Rocheteau and Wright (2005), there exist a continuum of submarkets, each identified by masses of sellers posting the same price $p \in \mathbf{P}$, with $\mathbf{P} \subset \mathbb{R}_+$ being the set of prices. After observing all the posted prices, each buyer chooses to enter one submarket that gives him the maximum surplus. Each seller can only produce for one buyer in each period. If a seller is visited by multiple buyers, he chooses one with equal probability. Let $n \leq N$ be the measure of active buyers in the DM and $n(p)$ be the market tightness in any submarket associated with p. In what follows we omit p as an argument in n. As before, the meeting rate for sellers is $\alpha(n)$ and $\alpha(n)/n$ for buyers. We seek a symmetric competitive search equilibrium, in which homogeneous buyers and sellers make the same optimal choice and are indifferent across all the submarkets. Without loss of generality, we can then focus on one submarket to solve for equilibrium, as in Rocheteau and Wright (2005). In equilibrium, the set of submarkets is complete so that no submarket could be created to make some buyers and sellers better off.

The buyer's DM value function is now

$$V_t^b(\tilde{p}, \tilde{n}, a) = \frac{\alpha(\tilde{n})}{\tilde{n}}(u - \tilde{p}) + W_t^b(a), \tag{14}$$

where $\tilde{n}$ and $\tilde{p}$ are the market tightness and posted price in a local market. From (1) and (14), the buyers' value function becomes

$$W_t^b(a) = \Sigma + (\varphi_t + \rho)a + \beta W_{t+1}^b(0) + \max_{\hat{a}, \tilde{p}, \tilde{n}} \left\{ \beta \frac{\alpha(\tilde{n})}{\tilde{n}}(u - \tilde{p}) + \left[\beta\left(\varphi_{t+1} + \rho\right) - \varphi_t\right]\hat{a} \right\}.$$

Let $\bar{V}^b \equiv \max_{p \in \mathbf{P}}\{\alpha(n)/n(u-p) - sp\}$ be the equilibrium expected utility of a buyer in the DM.[10] Note that we omit the time subscript to focus on stationary equilibrium. Taking $\bar{V}^b$ as given, a seller solves

$$\tilde{V}^s(\tilde{p}, \tilde{n}) = \max_{\tilde{p}, \tilde{n}} \alpha(\tilde{n})(\tilde{p} - c) \text{ st. } \tilde{V}^b(\tilde{p}, \tilde{n}) = \frac{\alpha(\tilde{n})}{\tilde{n}}(u - \tilde{p}) - s\tilde{p} \geq \bar{V}^b, \; \tilde{p} \leq (\varphi + \rho)a. \tag{15}$$

[9] For an extensive treatment of competitive search, see Wright et al. (2019).

[10] This is the market utility in McAfee (1993), Moen (1997), and Rocheteau and Wright (2005). It is the maximum expected utility buyers can get in any submarkets.

The constraint $\tilde{V}^b(\tilde{p},\tilde{n}) = \bar{V}^b$ determines the beliefs about market tightness n generated by p on an off-equilibrium path.

Given that prices are observed before buyers choose their asset holdings, we have $\tilde{p} = (\varphi + \rho)\, a$. Substituting $\tilde{p}$ from the constraint yields

$$\max_{\tilde{n}} \alpha(\tilde{n}) \left[\frac{\alpha(\tilde{n})\, u - \tilde{n}\bar{V}^b}{\alpha(\tilde{n}) + \tilde{n}s} - c \right]. \tag{16}$$

It is easy to show that the necessary condition of the above optimization problem is also sufficient. Using $\bar{V}^b$ and the constraint, we derive the seller's optimal price

$$\tilde{p}^c(s,\tilde{n}) = \frac{\alpha(\tilde{n}) \{[1-\varepsilon(\tilde{n})]\, u + \varepsilon(\tilde{n})\, c\} + \varepsilon(\tilde{n})\, \tilde{n}sc}{\alpha(\tilde{n}) + \varepsilon(\tilde{n})\, \tilde{n}s}, \tag{17}$$

where $\varepsilon(n) = \alpha'(n)n/\alpha(n)$ is the elasticity of the matching rate and $\varepsilon(n) < 1$.[11] In a symmetric equilibrium, the market tightness in different local markets is the same, i.e., $\tilde{n} = n$, and the optimal price is also the same across different local markets, i.e., $\tilde{p}^c(s,\tilde{n}) = p^c(s,n)$. The equilibrium n satisfies the free entry condition

$$\frac{\alpha(n)}{n}(u - p^c) - sp^c = \bar{V}^b \geq 0. \tag{18}$$

Equations (17) and (18) yield p^c and n as functions of the asset spread, and we equate the aggregate demand and supply of liquidity to determine s. The liquidity demand is $L^d(s) = n(s)p^c(s,n)$ and the supply $L^s(s) = (\varphi+\rho)A^s = (1+r)\rho A^s/(r-s)$. Similar to the bargaining case, let s^N be the cutoff spread value for full entry, $\bar{s}$ the spread that gives buyers zero surplus, and ρ^F the cutoff dividend value above which assets are priced fundamentally. Then, L^d and L^s are characterized by the following lemmas and illustrated in Figure 4.

Lemma 3 *There exist $\bar{s} \geq r$ and $s^N \leq \bar{s}$, such that: (i) for $s \leq s^N$, $\exists!$ L^d with $n = N$ and $dL^d/ds < 0$; (ii) for generic $s \in (s^N, \bar{s}]$, $\exists!$ L^d with $n < N$ and $dL^d/ds < 0$; (iii) for $s > \bar{s}$, $\nexists$ $n > 0$ and L^d is not well-defined.*

[11] Interestingly, if we have divisible goods $q \in \mathbb{R}_+$ and sellers posting (p,q), we can obtain an additional optimal condition $u'(q)/c'(q) - 1 = s/\alpha(n)$, as the liquidity premium in Lagos and Wright (2005) with money. Thus, when asset is priced at its fundamental value, we get the efficient quantity. Furthermore, solving this equation for s and substituting it into (17) gives $p^c = \{[1-\varepsilon(n)]\, u(q)c'(q) + \varepsilon(n)u'(q)c(q)\}/\{\varepsilon(n)\, u'(q) + [1-\varepsilon(n)]c'(q)\}$, the standard pricing equation in a monetary environment. With bargaining, $\varepsilon(n)$ is replaced by the buyer's bargaining power.

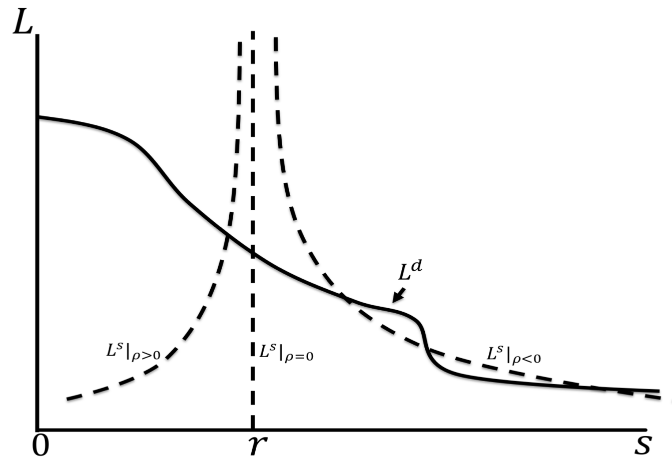

Figure 4: Supply and Demand of Liquidity with Competitive Search

Lemma 4 *There exists $\rho^F > 0$, such that: (i) For $\rho < 0$ $(r < s)$, L^s is convex and $dL^s/ds < 0$; (ii) for $\rho = 0$, L^s is perfectly elastic at $s = r$; (iii) for $\rho \in (0, \rho^F)$ $(0 < s < r)$, L^s is convex and $dL^s/ds > 0$; (iv) for $\rho \geq \rho^F$, L^s is perfectly elastic at $s = 0$.*

Note that with bargaining, the trading price in the DM is always equal to c, independent of the buyer's entry and the spread. However, with competitive search, p^c is a function of n and s, causing L^d to be highly nonlinear, and complicates the analysis for $\rho < 0$. Next, we are ready to define and characterize the competitive search equilibrium.

Definition 2 *A stationary competitive search equilibrium (SCE) is a list $\{p^c, s, n\}$ such that: (i) given $\tilde{n} = n$, p^c satisfies (17); (ii) $s > 0$ solves $L^d(s) = L^s(s)$ or $s = 0$; (iii) $\tilde{n} = n \leq N$ solves (18).*

Proposition 3 *There exists $\underline{\rho} < 0$, such that: (i) for $\rho \geq 0$, $\exists!$ SCE; (ii) for $\rho \in [\underline{\rho}, 0]$, $\exists$ SCE.*

Similar to the bargaining case, there is a unique equilibrium for $\rho \geq 0$. For $\rho < 0$, buyers face a similar trade-off between the probability of trade and the opportunity cost of holding assets, and there may exist multiple equilibria. We cannot prove that there are exactly two equilibria, since p^c changes with respect to n and s and L^d can be highly nonlinear under competitive search, while with bargaining, L^d is a straight line and p^b is the seller's reservation value, independent of the spread and the market tightness in the DM.

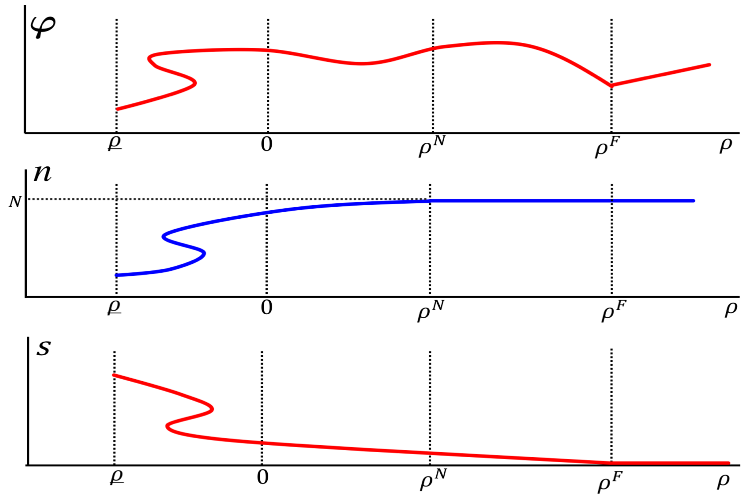

Figure 5: Competitive Search Equilibrium, $\rho^N > 0$

With competitive search, sellers post prices to direct the buyers' search behavior and serve as a coordinating device. Hence, in the monetary economy with competitive search in Rocheteau and Wright (2005) and Han et al. (2016), there is a unique equilibrium. However, while the cost of holding money is an exogenous policy variable in their economy, the cost of holding assets is endogenously determined in our model. For a given s, one can show that the seller's optimal price is unique and there is a unique optimal entry. Then, since s is endogenously determined by $L^s = L^d$, there may exist multiple equilibria with different s, each generating the same payoff for buyers and sellers, respectively. For example, one equilibrium may feature a high probability of trade and a large spread for carrying assets, while another has a low probability and a small cost of liquidity, both giving buyers the same equilibrium expected utility and making sellers indifferent between posting two prices. In this situation, competitive search still serves as a coordinating device given the spread, but there are equilibria featuring different s.

Like the bargaining case, we define the cutoff dividend value for full entry as ρ^N, solving $\bar{V}^b = 0$. As we discuss the comparative statics in the following, we focus on the case of $\rho^N > 0$, which is illustrated in Figure 5, and Figure 6 presents the case of $\rho^N < 0$. Similar to the bargaining case, the cutoffs of ρ^N, ρ^F, and $\underline{\rho}$ define intervals of ρ, featuring equilibria with different amount of liquidity.

For $\rho > \rho^F$, the total liquidity provided by assets is so abundant that all buyers enter the DM. The asset is priced at the fundamental value and the asset price φ is

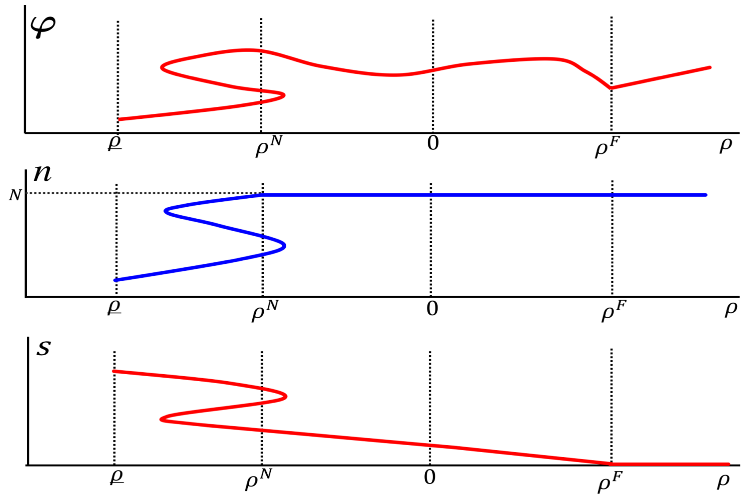

Figure 6: Competitive Search Equilibrium, $\rho^N < 0$

monotonically increasing with the dividend. As ρ decreases and becomes less than ρ^F but still greater than zero, the total liquidity is still enough to support full entry by buyers, and the asset carries a liquidity premium in its price. In this case, as ρ decreases, the total supply of liquidity L^s also decreases, and hence s increases, as one can see from Figure 4. The change of φ, however, is ambiguous for the following reason. On the one hand, a smaller ρ implies a lower liquidity supply. On the other hand, as ρ decreases, the cost of holding assets increases, implying a lower p^c in the DM, and the total liquidity demand also drops. Both L^s and L^d change in the same direction, and the change of φ due to decreasing ρ is unclear. With bargaining, we always have $p^B = c$ and the liquidity demand is a constant with full entry. Hence, as ρ decreases, the asset price increases monotonically to fill in the liquidity gap caused by a lower ρ.

As the dividend decreases even further, there is not enough liquidity to support full entry and some buyers drop out of the DM. The cost of holding assets increases, and the change in asset price is still ambiguous. For $\rho < 0$, p^c depends on both s and n, making it impossible to fully characterize L^d. Now both the "hot potato" and the "stinky fish" effect kick in. Under general parameter values, either one may be the dominant force, and the comparative statics of φ, n, and s are ambiguous. Finally, the DM shuts down when $\rho < \underline{\rho}$. The analysis is similar in the case of $\rho^N < 0$, and the adjustment on the extensive margin happens at a later time. In the following, Proposition 4 summarizes the effects of changing ρ in equilibria.

Proposition 4 *In the SCE: (i) for $\rho \geq \rho^F$, we have $\partial\varphi/\partial\rho > 0$, $n = N$, and $s = 0$; (ii) for $\rho \in [\rho^N, \rho^F)$ and $\rho \geq 0$, $n = N$, $\partial s/\partial\rho < 0$, and $\partial\varphi/\partial\rho$ is ambiguous; (iii) for $\rho \in (\underline{\rho}, \rho^N)$ and $\rho \geq 0$, $\partial n/\partial\rho > 0$, $\partial s/\partial\rho < 0$, and $\partial\varphi/\partial\rho$ is ambiguous; (iv) for $\rho < 0$, $\partial n/\partial\rho$, $\partial s/\partial\rho$, and $\partial\varphi/\partial\rho$ are ambiguous.*

4 Lotteries

Following Berentsen et al. (2002), we introduce lotteries in our non-convex economy with indivisible goods and study the effect of lotteries on equilibrium. Let $E = \mathbf{P} \times \{0, 1\}$ denote the space of trading events, and $\mathbf{W}$ the Borel σ-algebra. Define a lottery to be a probability measure ω on the measurable space $(E, \mathbf{W})$. We can write $\omega(p, q) = \omega_q(q)\omega_{p|q}(p)$ where $\omega_q(q)$ is the marginal probability measure of q and $\omega_{p|q}(p)$ the conditional probability measure of p on q. As shown in Berentsen et al. (2002), we can restrict attention to $\tau = \Pr\{q = 1\}$ and $1 - \tau = \Pr\{q = 0\}$, and $\omega_{p|0}(p) = \omega_{p|1}(p) = 1$. Randomization is only useful on q because $\mathbf{Q}$ is non-convex. Hence, $\tau \in [0, 1]$ is the probability that the good is produced and traded.

With lotteries, the generalized Nash bargaining problem becomes

$$\max_{p,\tau} (\tau u - p)^{\eta} (p - \tau c)^{1-\eta} \text{ st. } p \leq (\varphi + \rho) a,\ \tau \leq 1,\ \tau u \geq p, \text{ and } p \geq \tau c$$

and its solution is characterized by the following lemma.

Lemma 5 *The optimal solution to the bargaining problem with lotteries is*

$$\left(p^b, \tau^b\right) = \begin{cases} \left(\bar{p}^b, 1\right) & \text{if } (\varphi + \rho)a > \bar{p}^b \\ ((\varphi + \rho)a, 1) & \text{if } \underline{p}^b \leq (\varphi + \rho)a \leq \bar{p}^b \\ \left((\varphi + \rho)a, (\varphi + \rho)a/\underline{p}^b\right) & \text{if } c \leq (\varphi + \rho)a < \underline{p}^b \\ (0, 0) & \text{if } (\varphi + \rho)a < c \end{cases}$$

where $\bar{p}^b = (1 - \eta) u + \eta c$ and $\underline{p}^b = uc/(\eta u + (1 - \eta)c)$.

The buyer's CM value function now becomes

$$W_t^b(a) = \Sigma + (\varphi_t + \rho)a + \beta W_{t+1}^b (0) + \beta \max_{\hat{a}} v(\hat{a}),$$

where $v(\hat{a}) = (\tau^b u - p^b)\alpha(n)/n - s(\varphi_{t+1} + \rho)\hat{a}$. Proposition 5 characterizes equilibria.

Proposition 5 *There exists $\underline{\rho} < 0$, such that: (i) for $\rho \geq 0$ or $\rho = \underline{\rho} < 0$, $\exists!$ SBE with lotteries; (ii) for $\rho \in (\underline{\rho}, 0)$, $\exists$ SBE with lotteries; (iii) $p^b = \underline{p}^b$ and $\tau^b = 1$ in (i) and (ii).*

Since τ^b affects the expected surplus from the DM trade, i.e., $\tau^b(u - c)$, it is in the best interest of both buyers and sellers to negotiate the trading probability as high as possible. Hence, lotteries are never used in equilibrium and buyers bring just enough assets to achieve the maximum expected surplus from trade at $\tau^b = 1$. In equilibrium, $\underline{p}^b$ and τ^b do not change with respect to ρ, and the buyer's asset holding is not affected by the spread s. Compared to the case of bargaining without lotteries, we may get a continuum of equilibria for $\rho \in (\underline{\rho}, 0)$, since the trade-off between the probability of trade and the cost of carrying liquidity still exists. Lotteries do not lead to equilibrium uniqueness.

Finally, introducing lotteries to competitive search, the price posting problem becomes

$$\tilde{V}^s(p, n, \tau) = \max_{p,\tau,n} \alpha(n)(p - \tau c)$$
$$st.\ \tilde{V}^b(p, n, \tau) = \frac{\alpha(n)}{n}(\tau u - p) - sp \geq \bar{V}^b,\ p \leq (\varphi + \rho) a,\ \tau \leq 1.$$

The following proposition shows that lotteries are not used in competitive search equilibrium either, since sellers want to post $\tau^c = 1$ to maximize their expected profits and buyers' expected market utility.

Proposition 6 *There exists $\underline{\rho} < 0$, such that: (i) for $\rho \geq 0$, $\exists!$ SCE with lotteries; (ii) for $\rho \in [\underline{\rho}, 0]$, $\exists$ SCE with lotteries; (iii) $\tau^c = 1$ holds in (i) and (ii).*

5 Conclusion

This paper studies an economy where divisible real assets play an essential role in providing liquidity for transactions. Agents acquire assets and then use them as the medium of exchange to trade an indivisible good in a frictional market. The asset is valued for providing liquidity and its real dividend. The dividend value may be positive or negative. We consider two trading mechanisms, generalized Nash bargaining and price posting with competitive search. We allow free entry by the buyers of the indivisible goods, who are on the demand side of asset liquidity.

Under both trading mechanisms, we find that the equilibrium is unique if the dividend is positive, and there may exist multiple equilibria if $\rho < 0$ since buyers then face a trade-off between the opportunity cost of holding assets and the probability of trade. To compare with the equilibrium result in a monetary economy, the key difference is that the cost of carrying liquidity is endogenously determined by the supply and demand of assets, and the latter is affected by free entry.

Indivisibility matters, especially when the terms of trade in the goods market are determined by bargaining. Indivisibility affects the bargaining outcome because it isolates the price of goods from the cost of carrying assets, hence the dividend value, and the number of buyers. The bargained price gives sellers no surplus from trade. Introducing lotteries does not change the independence of the price, but sellers are able to extract a positive surplus. Price posting with competitive search serves as a coordinating device. It reestablishes the link between trading price and asset dividend as well as entry. Introducing lotteries again does not change the equilibrium results.

Since assets serve as the medium of exchange in this economy, the asset price always adjusts to create enough liquidity supply to satisfy the demand. Hence, depending on the different levels of liquidity abundance in the economy, the model predicts different relationships between asset price and asset dividend. While we have focused on stationary equilibrium, the model can easily be used to study asset price dynamics. We leave this for future research.

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Appendix

Proof of Lemma 1. Given buyers' participation constraint, $n = N$ if $\frac{\alpha(n)}{n}(u-c) - sc > 0$. Define $s^N = \frac{\alpha(N)}{N}\frac{u-c}{c}$, then given $s \leq s^N$, for all $n < N$, $\frac{\alpha(n)}{n}(u-c) - sc > 0$, contradiction. Hence $n = N$, and hence (i). Define $\bar{s} = \frac{u-c}{c}$, then $\forall n \in (0, N]$, for $s > \bar{s}$, $\nexists\ n$ st $\frac{\alpha(n)}{n}(u-c) - sc \geq 0$, hence (iii). For $s \in (s^N, \bar{s}]$, $\frac{\alpha(n)}{n}(u-c) - sc = 0$. Then $dn/ds = \frac{\partial[\alpha(n)/n]}{\partial n}\frac{c}{u-c} < 0$, therefore $dL^d/ds < 0$, and hence (ii). ■

Proof of Lemma 2. If assets are priced at the fundamental value, then all buyers participate in the DM and $s = 0$. Let $\rho^F = (1-\beta)c/A$. If $\rho \geq \rho^F$, then $\forall\ n$ st $(\varphi + \rho)A^s/n \geq (\varphi^F + \rho)A^s/n \geq \rho^F A/(1-\beta) = c$. The liquidity need for assets is satisfied and the marginal holders of assets only care about the store of value function of assets. Hence, $\varphi = \varphi^F$ and $s = 0$, hence (iv). If $\rho = 0$, the cost of holding assets is $s = r$, hence (ii). Otherwise, $\varphi = (1+s)\rho/(r-s)$, then substitute s into the liquidity supply and $L^s = (1+r)\rho A^s/(r-s)$, with $\partial L^s/\partial s = (1+r)\rho A^s/(r-s)^2$ and $\partial^2 L^s/\partial s^2 = 2(1+r)\rho A^s/(r-s)^3$. It is easy to check $\partial L^s/\partial s > 0$ and $\partial^2 L^s/\partial s^2 > 0$ for $\rho \in (0, \rho^F)$, i.e., $0 < s < r$, hence (iii) and for $\rho < 0$, $\partial L^s/\partial s < 0$ and $\partial^2 L^s/\partial s^2 > 0$. Hence, (i) holds. ■

Proof of Proposition 1. Figure 1 illustrates the liquidity demand and the liquidity supply with bargaining. There may exist a unique or multiple intersections of demand and supply, and we need to discuss different cases. For $\rho \geq \rho^F$, L^s and L^d don't have an intersection for all $s > 0$. $L^s \geq L^d$. Therefore sellers hold some assets too. The equilibrium is unique. For all $\rho < \rho^F$, all equilibria satisfy $L^s = L^d$ and the buyers' participation constraint. For $\rho > 0$, we have $dL^s/ds > 0$ and $dL^d/ds \leq 0$, hence the equilibrium is unique. For $\rho = 0$, the asset case is equivalent to the fiat money case with zero money growth rate, and we show the uniqueness in Han et al. (2016) proposition 3. Then rewrite the constraint we get $-\rho \leq \frac{n}{A^s}\left[\frac{\alpha(n)}{n}\beta(u-c) - (1-\beta)c\right] \equiv f(n)$. Notice $f''(n) < 0$, then $f(n)$ has a unique global maximum point on the support $[0, N]$. Now define $\underline{\rho} = -\max_{n\in[0,N]} f(n)$. For $\rho < \underline{\rho} \leq 0$, $f(n) < -\rho\ \forall n > 0$, then the buyers' participation constraint doesn't hold and the DM shuts down, $L^s = L^d$ will never hold. For $\rho = \underline{\rho}$, the equilibrium is unique because of the unique n which maximizes $f(n)$. Hence (i). For $\rho \in (\underline{\rho}, 0)$, there are two roots st $f(n) = -\rho$, call them n_l and n_h. With the lose of generality, let $n_l < n_h$. Then it is easy to show $n_l < N$, then $n = n_l$ which satisfies $f(n_l) = -\rho$ and $L^s = L^d$ is an equilibrium. We focus on the other root n_h. If

$n_h \geq N$, we have $f(N) \geq f(n_h) = -\rho$. Then $n = N$ and $L^s = L^d$ is the other equilibrium; otherwise, $n = n_h < N$ and $L^s = L^d$ is the other equilibrium. In sum, for $\rho \in (\underline{\rho}, 0)$, the two equilibria with $n = n_l$ and $n = \max\{n_h, N\}$. Hence (ii). ■

Proof of Proposition 2. For the unique equilibrium or the equilibrium with higher participation, we have shown that there exists a cutoff ρ^F such that $\rho^F A^s = (1-\beta)cN$. Then, $\forall \rho > (1-\beta)cN/A^s$, we have $\varphi = \varphi^F$, $n = N$, and $s = 0$; hence (i). Define ρ^N by $\tilde{V}^b(N) = 0$, then $\rho^N = -\frac{N}{A^s}\left[\frac{\alpha(N)}{N}\beta(u-c) - (1-\beta)c\right]$. For $\rho \in [\rho^N, \rho^F)$, $f(n) > -\rho$ $\forall$ $n < N$, hence $n = N$ is a possible candidate equilibrium. If $\rho < \rho^N$, all equilibria should satisfy $n < N$. For $\rho \in [\rho^N, \rho^F)$, $n = N$ and $\varphi = cN/A^s - \rho$, hence $\partial\varphi/\partial\rho < 0$ and $\partial s/\partial\rho < 0$; hence (ii). For $\rho \in [\underline{\rho}, \rho^N)$, because of the concavity of $f(n)$, $\partial s/\partial\rho < 0$ and thus $\partial\varphi/\partial\rho > 0$ and $\partial n/\partial\rho > 0$; hence (iii). For the equilibrium with lower participation and $\rho \in (\underline{\rho}, 0)$, because of the concavity of $f(n)$, we have $\partial\varphi/\partial\rho < 0$ and $\partial n/\partial\rho < 0$, and $\partial s/\partial\rho > 0$. ■

Proof of Lemma 3. To prove that L^d is a well-defined function for $s \leq \bar{s}$, it is sufficient to show $n > 0$ exists and is unique. Substituting p^c into (18) gives $\alpha\varepsilon(u-c)s + \alpha^2\varepsilon(u-c)/n = \alpha[(1-\varepsilon)u + \varepsilon c]s + \varepsilon ncs^2$. Define $h(n,s) = \alpha\varepsilon(u-c)s + \alpha^2\varepsilon(u-c)/n - \alpha[(1-\varepsilon)u + \varepsilon c]s - \varepsilon ncs^2$. Given any $n \in (0, N]$, $h(n,s) = 0$ is a quadratic function in s, which has two real solutions with opposite signs. The positive solution s_+, satisfying $h(n, s_+) = 0$, is an implicit function of n, $s_+(n)$. Let $s_+(0) = \lim_{n\to 0} s_+(n) < \infty$, and $s_+(0)$ is continuous on $[0, N]$. Define s^N by $h(N, s^N) = 0$ and $\bar{s} = \max_{n\in[0,N]} s_+(n)$. For $s < s^N$, $h(N, s) > 0$ hence $n = N$. Then $L^d = Np^c(N, s)$ is unique, and $dL^d/ds = Ndp^c(N,s)/ds < 0$, hence (i). For $s > \bar{s}$, $h(n,s) < 0$ $\forall n$, and the free-entry condition does not hold due to $\alpha(n)(u - p^c)/n - sp^c < 0$, hence (iii).

Regarding (ii), for $s \leq \bar{s}$, $h(n,s) = 0$ always holds for some $n > 0$, and L^d exists. To show that L^d is generically unique and monotone, consider $L^d = np^c$ and $dL^d/ds = \partial L^d/\partial s + (\partial L^d/\partial n)(\partial n/\partial s)$. Given $h(n,s) = 0$, we have $L^d = \alpha(n)nu/[\alpha(n) + sn]$, hence $\partial L^d/\partial s < 0$ and $\partial L^d/\partial n > 0$. Then, it is sufficient to show that n is generically unique and $\partial n/\partial s < 0$. We claim that although there might be multiple n which maximize $\tilde{V}^s(\tilde{n}, s)$, n is still unique and $\partial n/\partial s < 0$ for generic s. To see this, suppose $\tilde{V}^s(n_1, s) = \tilde{V}^s(n_2, s) = \max_{\tilde{n}} \tilde{V}^s(\tilde{n}, s)$ and $n_2 > n_1$. Then, n_1 is the minimum n maximizing $\tilde{V}^s(\tilde{n}, s)$, and $\tilde{V}^s(n_1, s) > \tilde{V}^s(\tilde{n}, s)$, $\forall \tilde{n} < n_1$. For $\epsilon > 0$ small enough, $\tilde{V}^s(n_1, s+\epsilon) > \tilde{V}^s(\tilde{n}, s+\epsilon)$

also holds for $n < n_1$ due to continuity. If $\partial^2\pi/\partial s\partial n < 0$, then $\tilde{V}^s(n_1, s+\epsilon) > \tilde{V}^s(n_2, s+\epsilon)$, and the global maximizer is a unique n in the neighborhood of n_1. Next, we need to show $\partial^2\tilde{V}^s/\partial s\partial n < 0$. Derive $\partial\tilde{V}^s/\partial\tilde{n}$ from (16),

$$\frac{\partial\tilde{V}^s}{\partial\tilde{n}} = \frac{(\alpha + s\tilde{n})\left[(u-c)\,\alpha' - sc\right] - s\,(1-\varepsilon)\left[(u-c)\,\alpha - s\tilde{n}c\right]}{(\alpha + s\tilde{n})^2/\alpha}.$$

Define $T(s) = (\alpha + s\tilde{n})[(u-c)\alpha' - sc] - s(1-\varepsilon)[(u-c)\alpha - s\tilde{n}c]$, and $T'(s) = \tilde{n}[(u-c)\alpha' - sc] - (\alpha + s\tilde{n})c - (1-\varepsilon)[(u-c)\alpha - s\tilde{n}c] + s\tilde{n}c(1-\varepsilon)$. Since $T_{\tilde{n}=n} = 0$, $\partial^2\pi/\partial s\partial n = T'(s)/[(\alpha + sn)^2/\alpha]$. With $\alpha(u-c) - snc > 0$, we have

$$T'_{\tilde{n}=n}(s) = \frac{-\left[\alpha\,(u-c) - snc\right](1-\varepsilon)\,\alpha - c\,(\alpha + sn)\,(\alpha + sn\varepsilon)}{\alpha + sn} < 0.$$

Therefore, $\partial^2\tilde{V}^s/\partial s\partial n < 0$ holds. In addition, $\arg\max_{\tilde{n}}\tilde{V}^s(\tilde{n}, s)$ might have more than one solution for some $s \geq s^{NC}$, but the set of such asset spreads has measure zero, hence (ii). Finally, we prove $\bar{s} \geq r$ by contradiction. Suppose $\bar{s} < r$, then for $s_1 = (r\varphi_1 - \rho_1)/(\varphi_1 + \rho_1) \in (\bar{s}, r)$, $\rho_1 > 0$ and $n_1 = 0$. Hence, $\varphi_1 = \varphi_1^F$ and $s_1 = 0$, contradicting $s_1 > \bar{s} > 0$. ■

Proof of Lemma 4. If assets are priced at the fundamental value, then all buyers participate in the DM and $s = 0$. Let $\rho^F = (1-\beta)p^c_{N,s=0}/A$. If $\rho \geq \rho^F$, the average asset holding $(\varphi + \rho)A^s/n \geq (\varphi^F + \rho)A^s/n \geq \rho^F A/(1-\beta) = p^c_{N,s=0}$. The liquidity need for assets is satisfied and the marginal holders of assets only care about the store of value function. Hence, $\varphi = \varphi^F$ and $s = 0$. If $\rho = 0$, the cost of holding assets is $s = r$. If $\rho < \rho^F$ and $\rho \neq 0$, substitute s into the liquidity supply and $L^s = (1+r)\rho A^s/(r-s)$, with $\partial L^s/\partial s = (1+r)\rho A^s/(r-s)^2$ and $\partial^2 L^s/\partial s^2 = 2(1+r)\rho A^s/(r-s)^3$. It is easy to check $\partial L^s/\partial s > 0$ and $\partial^2 L^s/\partial s^2 > 0$ for $\rho \in (0, \rho^F)$, and for $\rho < 0$, $\partial L^s/\partial s < 0$ and $\partial^2 L^s/\partial s^2 > 0$. ■

Proof of Proposition 3. Figure 4 illustrates the liquidity demand and the liquidity supply with competitive search. There may exist a unique or multiple intersections of demand and supply, and we need to discuss different cases. For $\rho \geq \rho^F$, a downward-sloping L^d and a perfectly elastic L^s ensure the existence and uniqueness of equilibrium s with $n = N$, hence (i). For $\rho = 0$, assets are equivalent to money with zero inflation, and the proof follows Proposition 5 in Han et al. (2016). For $\rho \in (0, \rho^F)$, L^d and L^s intersect once and there exists a unique equilibrium. For $\rho < 0$, $\bar{s} \geq r$ according to Lemma 3. If $\bar{s} = r$, $\nexists$ non-degenerate equilibrium; if $\bar{s} > r$, L^d and L^s may have more than one intersection, hence more than one candidate equilibrium. Given the equilibrium n being a function of s, we can rewrite the seller's problem (16) as

$$\max_s \alpha\left(n\left(s\right)\right)\left[\frac{\alpha\left(n\left(s\right)\right)u - n\left(s\right)\bar{V}^b}{\alpha\left(n\left(s\right)\right) + n\left(s\right)s} - c\right].$$

Given different values of s satisfying the first-order condition, there could be more than one s which maximize seller's profit. Hence $\nexists$ uniqueness in this region. Next is to show the existence of $\underline{\rho}$. If $\bar{s} = r$, $\underline{\rho} = 0$. Consider $\bar{s} > r$. For $s \in (r, \bar{s})$, $\rho < 0$, $\partial L^s/\partial\rho = (1+r)A^s/(r-s) < 0$, and L^d is constant. Hence, $\forall s \in (r, \bar{s})$, $\exists!\ \rho(n)$ such that $L^s(\rho) = L^d$, and define $\underline{\rho} = \min_{s\in[r,\bar{s}]} \rho(s) < 0$. For $\rho < \underline{\rho}$, $L^s > L^d$, and there exists no equilibrium. ■

Proof of Proposition 4. We know ρ^F satisfying $\rho^F A^s = (1-\beta)\,N[1-\varepsilon(N)]u + (1-\beta)\,N\varepsilon(N)c$. For $\rho \geq \rho^F$, we have $s = 0$, implying $\varphi = \varphi^F$ and $n = N$; hence (i). And we know $\rho^N = (1-\beta)\,p^c(s^N, N) - \beta[u - p^c(s^N, N)]\alpha\left(N\right)/N$, for $\rho \in (\rho^N, \rho^F)$, we have $n = N$ and $\partial s/\partial\rho < 0$ because L^d is downward slopping and L^s shifts upward with increasing ρ. However $\partial\varphi/\partial\rho$ is ambiguous. Hence (ii). For $\rho \in [\underline{\rho}, \rho^N)$ and $\rho \geq 0$, $\partial s/\partial\rho < 0$ and $\partial n/\partial\rho > 0$ because L^d is downward slopping and L^s shifts upward with increasing ρ. However $\partial\varphi/\partial\rho$ is ambiguous. hence (iii). For $\rho < 0$, $\partial s/\partial\rho$, $\partial n/\partial\rho$, and $\partial\varphi/\partial\rho$ are ambiguous because of the multiplicity, hence (iv). ■

Proof of Lemma 5. Using λ_1 and λ_2 for the multipliers on the asset constraint and the lotteries constraint gives the following Kuhn-Tucker conditions.

$$\begin{aligned}
0 &= -\eta\left(\tau u - p\right)^{\eta-1}\left(p - \tau c\right)^{1-\eta} + \left(1-\eta\right)\left(\tau u - p\right)^{\eta}\left(p - \tau c\right)^{-\eta} - \lambda_1 && (19)\\
0 &= \eta u\left(\tau u - p\right)^{\eta-1}\left(p - \tau c\right)^{1-\eta} - c\left(1-\eta\right)\left(\tau u - p\right)^{\eta}\left(p - \tau c\right)^{-\eta} - \lambda_2 && (20)\\
0 &= \lambda_1\left[(\varphi+\rho)a - p\right]\\
0 &= \lambda_2\left(1-\tau\right).
\end{aligned}$$

It is straightforward to check that if $\lambda_1 = 0$, $p = \tau^b\bar{p}^b$. Substituting this into (20) implies $\lambda_2 > 0$, and hence $\tau^b = 1$. In order to support $\tau^b = 1$, buyer needs to bring enough asset to the DM trade, i.e. $(\varphi+\rho)a > \bar{p}^b$. On the other hand, if $\lambda_2 = 0$, $\tau^b = p^b/\underline{p}^b$.

Substituting this into (19) implies $\lambda_1 > 0$ and $p^b = (\varphi + \rho)a$. In order to satisfy $\tau^b < 1$, we need $(\varphi + \rho)a < \underline{p}^b$. If both λ_1 and λ_2 are greater than zero, $p^b = (\varphi + \rho)a$ and $\tau^b = 1$. $\lambda_1 > 0$ implies $(\varphi + \rho)a < \bar{p}^b$, and $\lambda_2 > 0$ implies $(\varphi + \rho)a > \underline{p}^b$. Finally, the seller certainly does not trade if he meets a buyer with $(\varphi + \rho)a < c$. ■

Proof of Proposition 5. First, buyers do not want to bring $(\varphi_{t+1} + \rho)\hat{a} > \bar{p}^b$, since additional assets do not affect the surplus from trade. Second, they do not bring $(\varphi_{t+1} + \rho)\hat{a} < c$, for no trade. Next, for $(\varphi_{t+1} + \rho)\hat{a} \in (\underline{p}^b, \bar{p}^b)$, $v'(\hat{a}) = -(\varphi_{t+1} + \rho)[s + \alpha(n)/n] < 0$, and buyers want to choose $(\varphi_{t+1} + \rho)\hat{a} = \underline{p}^b$. For $(\varphi_{t+1} + \rho)\hat{a} \in (c, \underline{p}^b)$, $v'(\hat{a}) = (\varphi_{t+1} + \rho)\left[\alpha(n)\eta(u - c)/nc - s\right]$, and the sign of $v'(\hat{a})$ depends on the value of the spread s. Since $\alpha(n)(u - \underline{p}^b)/n - s\underline{p}^b = \underline{p}^b[\alpha(n)\eta(u-c)/nc - s]$, $v'(\hat{a})$ shares the same sign as $\alpha(n)(u - \underline{p}^b)/n - s\underline{p}^b$. Suppose $v'(\hat{a}) < 0$, buyers choose $\tau^b = 0$ and there is no equilibrium with an open DM. If $v'(\hat{a}) > 0$, buyers of measure n in the DM choose $(\varphi_{t+1} + \rho)\hat{a} = \underline{p}^b$. The cutoff spread satisfying $v'(\hat{a}) = 0$ is given by $\alpha(n)(u - \underline{p}^b)/n - s\underline{p}^b = 0$, which is equivalent to the participation constraint $n[\alpha(n)\beta(u - \underline{p}^b)/n - (1-\beta)\underline{p}^b]/A^s = g(n) \geq -\rho$. Since $g''(n) < 0$, let $\underline{\rho} = -\max_n g(n)$, $\rho^F = (1-\beta)\underline{p}^b/A$, and $\rho^N = [(1-\beta)\underline{p}^b - \beta\alpha(N)(u - \underline{p}^b)/N]/A$. For $\rho \geq \underline{\rho}$, all equilibria feature $p^b = \underline{p}^b$ and $\tau^b = 1$. If $\rho \geq \rho^F$, then $\varphi = \varphi^F$; otherwise $\varphi > \varphi^F$. If $\rho \geq \rho^N$, then $n = N$; otherwise $n < N$. The rest of the proof on equilibrium stability follows directly from Proposition 1. ■

Proof of Proposition 6. We need to check that sellers always post $\tau^c = 1$ and the rest of the proof follows Proposition 3. Let λ be the multiplier for τ, and the FOCs are

$$
\begin{aligned}
0 &= \varepsilon(\tilde{n})(p - \tau c) - \frac{\alpha(\tilde{n})[1 - \varepsilon(\tilde{n})](\tau u - p)}{\alpha(\tilde{n}) + \tilde{n}s}, &(21)\\
0 &= \tau\left[\frac{\alpha^2(\tilde{n})u}{\alpha(\tilde{n}) + \tilde{n}s} - \alpha(\tilde{n})c - \lambda\right], &(22)\\
0 &= \lambda(1 - \tau).
\end{aligned}
$$

Given the buyer's optimal participation $\tilde{n} = n$ and (21), we have

$$p^c = \frac{\alpha(n)\{[1 - \varepsilon(n)]\tau u + \varepsilon(n)\tau c\} + \varepsilon(n)ns\tau c}{\alpha(n) + \varepsilon(n)ns}.$$

Solve for λ from (22), and we need $\lambda = \alpha(n)(u - c) - cns > 0$ to assure $\tau^c = 1$. Since $p^c/\tau > c\ \forall\tau$, $\alpha(n)(u - c) - cns > \alpha(n)(u - p^c/\tau) - nsp^c/\tau \geq 0$. The last inequality is the buyer's participation constraint in the DM, which holds if $\rho \geq \underline{\rho}$ and $n > 0$. ■