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Credit, Money and Asset Equilibria with Indivisible Goods

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Abstract

We study the trade of indivisible goods using credit, divisible money and divisible assets in a frictional market. We show how indivisibility matters for equilibria. Bargaining generates a price that is not linked to nominal interest rates, dividend value of the asset, or the number of active buyers. To reestablish this connection, we consider price posting with competitive search. We provide conditions under which stationary equilibrium exists. With bargaining, we find that for negative dividend value on the asset, multiple equilibria occur. Otherwise, in all possible combinations of liquidity and price mechanisms the equilibrium is unique or generically unique.

JEL: D51, E40.

Keywords: Nash Bargaining; Competitive Search; Indivisibility; Multiplicity; Uniqueness.

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1 Introduction

Use of the New Monetarist framework has seen an increasing growth in popularity with numerous applications in areas such as finance, payment systems, and monetary analysis. Lagos et al. (2016) cite numerous papers using this framework. A standard version of the model, based on Lagos and Wright (2005), assumes that agents trade divisible goods using divisible money as the medium of exchange. The literature emerged from the original search based models of Kiyotaki and Wright (1989, 1993) with indivisible goods and money and further from Shi (1995) and Trejos and Wright (1995) using divisible goods and indivisible money.[1] Due to the indivisibility of the medium of exchange these models have a trivial exogenous distribution of money. However, with divisible money the distribution of money holdings is endogenous and non-degenerate. One must resort to computation as in Molico (2006) or force a degenerate distribution with special assumptions as in Lagos and Wright (2005) and Shi (1997).

In this paper we explore the consequences of having indivisible goods traded with divisible assets. Effectively, this is the reverse of the divisible good, indivisible money model used in Shi-Trejos-Wright. In that framework, agents cannot accumulate more than one unit of money. Here, we assume that buyers only want to consume one unit of the indivisible good but can hold any amount of money.[2]

We compare equilibrium allocations under different scenarios. First, we assume sellers can extend credit in the frictional market to be repaid by buyers in the subsequent centralized market. Second, we consider a monetary economy. Third, we consider an economy where a real asset is used as a medium of exchange. Unlike the monetary model, the asset is in fixed total supply and it bears an exogenous dividend which could be positive or negative. The reason to consider real assets is that when credit is imperfect, money is not the only object that can serve in the medium of exchange role.

We assume terms of trade determined by generalized Nash bargaining or price posting with competitive search.[3] The reason why we study different pricing mechanisms is that,

[1] In Shi (1995) and Trejos and Wright (1995), terms of trade are settled by bargaining. With divisible goods and indivisible money, Curtis and Wright (2004) study price posting with random search, while Julien et al. (2008) study auctions and price posting with directed search.

[2] Many goods have an indivisible component. The indivisibility could be determined by natural aspects, firms packaging strategy or a minimal assembly required to make the good operational. For instance, housing, cars, boats and other durable goods can be considered indivisible. The assumption of divisible goods can be re-interpreted as allowing buyers to purchase many units of an item at its lowest indivisible denomination.

[3] The competitive search framework we use is based on Moen (1997) and Mortensen and Wright (2002).

due to goods' indivisibility, each mechanism produces different results. With indivisible goods, no adjustment can take place through the intensive margin and the available surplus is fixed. In particular, with money/asset, the bargained price does not depend on nominal interest or return on the asset, while posted prices have this dependence.

We show uniqueness of equilibrium in the pure credit economy. Because no medium of exchange is needed, and hence there is no direct cost associated with liquidity, all potential buyers participate in the decentralized market and the solution is akin to the optimal solution in a monetary economy with divisible goods. The allocations in the money and asset economies differ from those in the credit economy. With bargaining, buyers can commit to bringing the lowest amount of money/asset needed to make sellers indifferent between trading or not trading. This is driven by no adjustment through the intensive margin. The solution is akin to a take-it-or-leave-it offer and buyers can extract the whole surplus. With price posting and with lotteries, sellers are able to extract some of the surplus. With price posting, sellers post the price before buyers enter the decentralized market and with lotteries comes the threat of not delivering the good.

In the monetary economy we show generic uniqueness and existence of a monetary equilibrium, as long as the nominal interest rate is not too high. The exogenous nominal rate determines the cost of holding money. Hence, this cost acts as an entry cost into the decentralized market. When the nominal rate is large the congestion induced by the matching technology reduces the number of buyers until monetary equilibria cease. The threshold for the nominal interest rate differs between the bargaining and the competitive search environment. The reason is that the equilibrium price depends on the number of participating buyers under competitive search while it does not under bargaining.

Under bargaining, for a low range of nominal interest rate, all potential buyers participate in the decentralized market. Money is then superneutral in the sense that it does not affect the real price, equal to sellers' cost, and the number of trades. However, for a higher range of nominal interest rates, not all buyers participate, and the expected number of trades is affected by increase in inflation. We show that using lotteries does not overturn the superneutrality.

The price determination of the indivisible good in the asset economy is similar to the case with money. With bargaining, the buyer is able to extract the full surplus from trade, while with competitive search the price of the indivisible good is a function of

For the use of competitive search in monetary models see Rocheteau and Wright (2005) and Lagos and Rocheteau (2007).

the participation rate and the cost of holding the asset. However, what is important is the dividend value of assets, which can be positive or negative. Under competitive search, the dividend also has an indirect effect on the price of the indivisible good through participation.

For high enough values of the dividend, all buyers participate in the decentralized market, with unique equilibrium under bargaining and competitive search. When the dividend is low, even negative, the matching congestion effect creates negative net expected benefit if all potential buyers participate. The cost of carrying the asset for trade is indirectly determined by the liquidity premium on the asset price which depends on the dividend value and the number of active buyers. In bargaining and competitive search, a higher number of active buyers means a lower probability of trade and a lower cost of carrying the asset. This generates multiplicity of equilibria. With bargaining, there are exactly two equilibria. One with large number of buyers participating, and the other with small number of buyers participating.

The literature on divisible money with indivisible goods includes Green and Zhou (1998) who consider price posting, but in a random rather than competitive search environment. Indivisible goods with posting lead to indeterminacy due to strong coordination effect. Jean et al. (2010) reconsider Green and Zhou (1988) using the Lagos and Wright (2005) framework and show that the indeterminacy result remains. We show that the coordination effect disappears with bargaining, and with competitive search. With bargaining, money/asset acts as a commitment to not pay more than what buyer brings. With competitive search, candidates from the continuum of Green and Zhou equilibria can be eliminated by sellers posting attractive terms of trade. Buyers respond to posting because they can direct their search. Galenianos and Kircher (2008) consider a model with terms of trade determined by second price auctions and characterize the equilibrium distribution of money holding. Liu, Wang and Wright (2015) consider terms of trade determined as in Burdett and Judd (1983) and focus on money and credit as competing payment instruments. Rabinovich (2015) studies commodity money with indivisible goods. All of this literature use a random search while we also use competitive search.

The paper is organized as follows. In Section 2 we describe the environment. In Section 3 we consider a pure credit economy with an exogenous credit constraint. In Section 4 we study a standard monetary economy. In Section 5 we consider an asset economy. Section 6 includes the study of lotteries and a conclusion follows.

2 Environment

The environment is based on the alternating markets framework of Rocheteau and Wright (2005).[4] Time is discrete and goes on forever. A continuum of buyers and sellers, with measures N and 1, live forever. In each period, all agents participate in two markets consecutively. Agents discount between periods with factor $\beta \in (0,1)$, but not across markets within a period, and $r = 1/\beta - 1$ is the discount rate. The first market to open is a decentralized market (DM), and the second is a frictionless centralized market (CM). Both buyers and sellers consume a divisible good in the CM, while only buyers consume an indivisible good in the DM.

Buyers' preferences within a period are given by $U(x_t) - h_t + u\mathbf{1}$, where x_t is CM consumption, h_t is CM labor, u is DM utility from consuming the indivisible good, and $\mathbf{1}$ is an indicator function giving 1 if trade occurs and 0 otherwise. Sellers' preferences are $U(x_t) - h_t - c\mathbf{1}$ with DM good produced at constant cost c. We assume $u > c$. Let x_t be the CM numeraire. We assume that x_t is produced one-to-one from labor h_t.

Trade in the DM implies a price and quantity bundle $(p, q) \in \mathcal{P} \times Q$ where $\mathcal{P} = \{0 \leq p \leq L\}$ and $Q = \{0, 1\}$. L represents the available liquidity in the economy, and $L = D$, an exogenous credit constraint in the credit economy. In the monetary economy, $L = \phi m$ represents the buyer's real money balance, and in the asset economy $L = (\varphi + \rho)a$, with ρ being the real dividend.

In the DM, meetings occur according to a general meeting technology which is assumed homogeneous of degree one. Given the buyer-seller ratio $n \leq N$, which is also the measure of participating buyers in the DM, the meeting rate for sellers is $\alpha(n)$, and $\alpha(n)/n$ for buyers. Assume $\alpha' > 0$, $\alpha'' < 0$, $\alpha(0) = 0$, $\lim_{n\to\infty} \alpha(n) = 1$, and $\lim_{n\to 0} \alpha'(n) = 1$.

3 Credit

Consider an economy in which commitment is feasible. Agents are not anonymous, record keeping and punishment devices are available. In this environment there is no role for money. Rather, sellers in the DM produce for buyers with the buyers promising to deliver

[4] The original alternating markets framework by Lagos and Wright (2005) has agents receiving a preference shock in the CM revealing whether they will be a buyer or a seller in the DM. In Rocheteau and Wright (2005), buyers are always buyers and sellers are always sellers. However, as discussed in Lagos and Rocheteau (2005) the difference between the frameworks is immaterial. All our results hold for both frameworks.

x_t in the subsequent CM. We assume an exogenous credit constraint $p \leq D$, where p is the real price and $D > 0$.

Buyers in the CM obtain

$$W_t^b(d) = \max_{x,h} \left\{ U(x) - h + \beta V_{t+1}^b \right\} \text{ s.t. } x = h - d, \tag{1}$$

where d is one's debt coming out of the DM, i.e. $d = p$ if a purchase and $d = 0$ otherwise. Buyers participate in the DM if $V_{t+1}^b \geq 0$. Using the budget constraint to eliminate h and solving for optimal x^* yields $W_t^b(d) = \Sigma + d + \beta V_{t+1}^b$ with $\Sigma = U(x^*) - x^*$. Sellers in the CM get

$$W_t^s(d) = \max_{x,h} \left\{ U(x) - h + \beta V_{t+1}^s \right\} \text{ s.t. } x = h + d,$$

where $d = -p$ if a sale and $d = 0$ otherwise. This simplifies to $W_t^s(p) = \Sigma - d + \beta V_{t+1}^s$. Sellers participate in DM if $V_{t+1}^s > 0$. The buyer's payoff in the DM is

$$V_t^b = \frac{\alpha(N)}{N} \left[u + W_t^b(p) \right] + \left[1 - \frac{\alpha(N)}{N} \right] W_t^b(0).$$

A buyer that trades obtains credit p, to be paid in the next CM, and gets utility u from DM consumption. Simplifying,

$$V_t^b = W_t^b(0) + \frac{\alpha(N)}{N}(u - p). \tag{2}$$

Similarly for sellers, $V_t^s = W_t^s(0) + \alpha(N)(p - c)$.

3.1 Bargaining

Upon meeting, a buyer and a seller solve the generalized Nash bargaining problem

$$\max_p (u - p)^\eta (p - c)^{1-\eta} \text{ s.t. } p \leq D.$$

Proposition 1 *In the model with credit and bargaining, $\exists!$ stationary equilibrium (SE) if $D \geq c$, characterized by*

$$p^B = \begin{cases} \bar{p}^B & \text{if } D > \bar{p}^B \\ D & \text{if } D \leq \bar{p}^B, \end{cases}$$

where $\bar{p}^B = (1 - \eta)u + \eta c$.

Proof. All Proofs are in Appendix. ■

Note that all buyers are active in the DM under credit since using credit is costless and $(u - p^B)\alpha(N)/N > 0$, for all attainable levels of p^B. As will be demonstrated, introducing money or assets as a medium of exchange in the DM can result in $n < N$ active buyers.

3.2 Competitive Search

We study competitive search equilibrium with price posting. As in Moen (1997), instead of a single DM, there exist a continuum of submarkets, each identified by masses of sellers posting the same terms of trade. Sellers post DM prices before buyers enter the DM. All sellers commit to their posted prices. After observing all the posted prices, each buyer chooses the one that gives him the maximum surplus. Each seller can only produce for one buyer in each period. If a seller is visited by multiple buyers, he chooses one with equal probability. Let n represent the expected queue length for any seller in a submarket offering price p. The meeting rates now depend on queue length induced by price, instead of the aggregate N. As before, the meeting rate for sellers is $\alpha(n)$, and $\alpha(n)/n$ for buyers in the submarket featuring p. By posting a lower price, a seller attracts more buyers and increases his trading probability.

Buyers' payoff in the CM is

$$W_t^b(d) = \Sigma + d + \beta \max_{\hat{p},n} \left\{ \frac{\alpha(n)}{n} (u - \hat{p}) + W_{t+1}^b(0) \right\},$$

where $\hat{p}$ represents the price seller gets in $t+1$. The seller's payoff in the DM is

$$V_t^s(p) = W_t^s(0) + \max_{p,n} \{\alpha(n)(p - c)\}.$$

Let Ω be the equilibrium expected utility of a buyer in the DM. To attract queue length n, sellers must offer price p satisfying $(u - p)\alpha(n)/n = \Omega$. A seller solves

$$\max_{p,n} \alpha(n)(p - c) \text{ s.t. } \frac{\alpha(n)}{n}(u - p) = \Omega, \, p \leq D.$$

Solve for p from the buyers' participation constraint, and substitute into the seller's objective function, to get

$$\max_{n} \alpha(n) \left[u - c - \frac{n\Omega}{\alpha(n)} \right] \text{ s.t. } u - \frac{n\Omega}{\alpha(n)} \leq D.$$

Proposition 2 *In the model with credit and competitive search,* $\exists!$ *symmetric SE if* $D \geq c$*, characterized by*

$$p^c = \begin{cases} \bar{p}^c & \text{if } D > \bar{p}^c \\ D & \text{if } D \leq \bar{p}^c \end{cases}$$

where $\bar{p}^c = [1 - \varepsilon(n)]u + \varepsilon(n)c$*, and* $n = N$*.*

As is standard, this result is identical to the case with bargaining when $\varepsilon(N) = \eta$ (Hosios (1990)). Similar to bargaining, $(u - p^c)\alpha(N)/N > 0$ for all p^c, implying that all buyers are active in the DM.

4 Money

Now assume agents in the DM cannot commit and there are no enforcement or punishment mechanisms. Therefore, buyers must pay sellers with cash in the DM. Let M_t^s be the money supply per buyer at time t, with $M_t^s = \gamma M_{t-1}^s$ and the growth rate of money, γ, is constant. Changes in M^s occur in the CM via lump-sum transfers (taxes) if $\gamma > 1$ ($\gamma < 1$). Nominal interest rate is given by the Fisher equation $1 + i = \gamma/\beta$ and we assume $\gamma > \beta$. The Friedman rule is the limiting case $i \to 0$. Define ϕ_t as the CM price of money in terms of x_t, and $1/\phi_t$ as the nominal price level. In stationary equilibrium, $\phi_t/\phi_{t+1} = \gamma$. Since there is a cost of carrying money, which may not be covered by the buyer's surplus from DM trade, we allow endogenous participation by buyers and let n denote the ratio of active buyers to sellers in the DM.

Buyers with money holding m in the CM solve

$$W_t^b(m) = \max_{x,h,\hat{m}} \left\{ U(x) - h + \beta V_{t+1}^b(\hat{m}) \right\} \text{ s.t. } x = \phi_t(m + T) + h - \phi_t \hat{m}, \tag{3}$$

where $\hat{m}$ is the money holding carried to the next DM, and T represents net transfers from the government only given to buyers. Eliminating h from the budget equation,

$$W_t^b(m) = \Sigma + \phi_t(m + T) + \max_{\hat{m}} \left\{ \beta V_{t+1}^b(\hat{m}) - \phi_t \hat{m} \right\}. \tag{4}$$

Sellers do not bring money into the DM. Thus,

$$W_t^s(m) = \Sigma + \phi_t m + \beta V_{t+1}^s \tag{5}$$

represents their CM value function.

Buyers' payoff in the DM is

$$V_t^b(m) = \frac{\alpha(n)}{n}\left[u + W_t^b\left(m - \frac{p}{\phi_t}\right)\right] + \left[1 - \frac{\alpha(n)}{n}\right] W_t^b(m), \tag{6}$$

where n represents the number of active buyers in the DM. If a buyer gets to trade, he pays p and gets u. Linearity, $\partial W_t^b/\partial m = \phi_t$, allows us to write

$$\begin{aligned} V_t^b(m) &= \frac{\alpha(n)}{n}(u - p) + W_t^b(m), \\ V_t^s &= \alpha(n)(p - c) + W_t^s(0). \end{aligned}$$

4.1 Bargaining

The generalized Nash problem is

$$\max_p (u - p)^\eta (p - c)^{1-\eta} \text{ s.t. } p \leq \phi m,\ u - p \geq 0,\ p - c \geq 0. \tag{7}$$

As is standard when $\gamma > \beta$, the feasibility constraint, $p \leq \phi m$, binds and $c \leq \phi m \leq \bar{p}^B$, where $\bar{p}^B = (1 - \eta)u + \eta c$ as in Proposition 1. Any negotiated price $p^B \in [c, \bar{p}^B]$ is a potential bargaining solution. Substituting V_{t+1}^b into W_t^b yields the following CM value function

$$W_t^b(m) = \Sigma + \phi_t(m + T) + \beta W_{t+1}^b(0) + \max_{\hat{m} \in [\underline{m}, \bar{m}]} \beta \left\{\frac{\alpha(n)}{n}\left(u - \phi_{t+1}\hat{m}\right) - i\phi_{t+1}\hat{m}\right\}, \tag{8}$$

where $\underline{m} = \frac{c}{\phi_{t+1}}$ and $\bar{m} = \frac{\bar{p}^B}{\phi_{t+1}}$. Since buyer's surplus decreases in $\hat{m}$, optimal money holding decision in (8) reduces to $\phi_{t+1}\hat{m} = c$. A buyer can effectively commit to not paying more than $p^B = \phi_{t+1}\hat{m}$. Bringing $\phi_{t+1}\hat{m} \geq \bar{p}^B$, yields $\eta(u - c)$ as buyer's surplus from trade, which is less than $u - c$, the surplus a buyer gets by bringing exactly $\phi_{t+1}\hat{m} = c$. The solution is akin to buyers making a take-it-or-leave-it offer to sellers in pairwise meetings.

Finally, we need to make sure that the buyer's surplus from trade in the DM covers the cost of carrying money. Define

$$v_n(\phi_{t+1}\hat{m}) = \frac{\alpha(n)}{n}\left(u - \phi_{t+1}\hat{m}\right) - i\phi_{t+1}\hat{m}. \tag{9}$$

The buyer's free entry condition $v_n(c) = 0$ implies

$$i = \frac{\alpha(n)}{n} \frac{(u-c)}{c} = \Psi(n). \tag{10}$$

The matching rate $\alpha(n)/n$ is decreasing in n, and so is $\Psi(n)$. Having fewer active buyers in the DM, reduces congestion and increases the marginal gain of entering the DM. Given i, (10) uniquely determines the measure of active buyers in the DM, n^*. Define $\bar{\imath}^N = \Psi(N)$ and $\bar{\imath}^B = (u-c)/c$. We can characterize equilibrium with the following proposition:

Proposition 3 *In the model with money and bargaining: (i) For $i \leq \bar{\imath}^N$, $\exists!$ stationary monetary equilibrium (SME) with $n^* = N$; (ii) for $i \in (\bar{\imath}^N, \bar{\imath}^B)$, $\exists!$ SME with $n^* < N$; (iii) for $i \geq \bar{\imath}^B$, $\nexists$ SME.*

Real balances in equilibrium only depend on u, c, and not bargaining power η or the nominal rate i. In this environment, buyers move first by choosing money balances. Then, buyers can commit to bringing the lowest level of real balances acceptable for trade. The nominal interest rate has no effect on the DM real price, buyer's real balances, or the real value of money. For $i \leq \bar{\imath}^N$, all buyers participate in the DM and the total output is not affected by i, either. Therefore, money is superneutral in the model with bargaining for small nominal interest rates.

This result differs from most of the New Monetarist literature, which generally features neutrality of money but real allocations are affected by changes in inflation. The generalized Nash bargaining mechanism determines the buyer's share of surplus according to exogenous bargaining power, which then determines the unique optimal real balance. Monetary variables do not play a role in the determination of real variables, but only affect the price of money ϕ.

When it is costless to carry money to the DM, i.e. $i = 0$, the monetary economy is comparable to the credit economy in Section 3.1, but with different price in the DM. When $i = 0$, buyers still choose to carry just enough real balance to cover the seller's reservation price c, so as to maximize their surplus from trade. As shown in Proposition 1, the equilibrium price with credit is almost always higher than the seller's reservation price. This is because when facing an exogenous credit constraint, buyers do not have the power to effectively commit to paying c ex ante.

4.2 Competitive Search

The next step is to study the implications of competitive search. The buyer's DM value function is now

$$V_t^b(p,m) = \frac{\alpha(n)}{n}(u-p) + W_t^b(m), \tag{11}$$

where p is the price posted by the buyer's chosen seller. From (4) and (11), buyers' value is

$$W_t^b(m) = \Sigma + \phi_t(m+T) + \beta W_{t+1}^b(0) + \max_{\hat{m},p,n}\beta\left\{\frac{\alpha(n)}{n}(u-p) - i\phi_{t+1}\hat{m}\right\}. \tag{12}$$

Since sellers post p before buyers choose their money holdings, $\phi_{t+1}\hat{m} = p$ as long as $i > 0$. Let Ω again be the equilibrium expected utility of a buyer in the DM. Sellers maximize

$$\max_{p,n} \pi(n) = \alpha(n)(p-c) \text{ s.t. } \frac{\alpha(n)}{n}(u-p) - ip \geq \Omega, \tag{13}$$

or

$$\max_{n} \pi(n) = \alpha(n)\left[\frac{\alpha(n)u - n\Omega}{\alpha(n)+ni} - c\right]. \tag{14}$$

In equilibrium, n^* is consistent with the free entry condition

$$\frac{\alpha(n^*)}{n^*}(u-p^c) - ip^c \geq 0, \tag{15}$$

and p^c is the seller's optimal price

$$p^c = \frac{\alpha(n^*)\{[1-\varepsilon(n^*)]u + \varepsilon(n^*)c\} + \varepsilon(n^*)n^*ic}{\alpha(n^*) + \varepsilon(n^*)n^*i}. \tag{16}$$

Unlike bargaining, p^c depends on i and n, the measure of active buyers in the market.

We follow Gu and Wright (2016) in establishing the existence and uniqueness of monetary equilibrium. Define the aggregate demand of liquidity, $L^d = n^*p^c$, with n^* and p^c both depending on i. Monetary equilibrium is then characterized by the intersection of L^d and the relevant supply curve, which is horizontal at the exogenous policy variable, i. The nominal interest rate is the price of holding liquidity. It determines the equilibrium quantity via L^d, which is characterized by the following lemma:

Lemma 1 *There exist i^N and $\bar{\imath}^C$ with $i^N < \bar{\imath}^C$, such that: (i) for $i < i^N$, $\exists!$ L^d with $n^* = N$ and $dL^d/di < 0$; (ii) for generic $i \in [i^N, \bar{\imath}^C]$, $\exists!$ L^d with $n^* \leq N$ and $dL^d/di < 0$;*

(iii) for $i > \bar{\imath}^C$, $\nexists\ n^* > 0$ and L^d is not well-defined.

Now we are ready to characterize symmetric monetary equilibrium, where all sellers post the same price and buyers visit each seller with the same probability.

Proposition 4 *In the model with money and competitive search: (i) for $i < i^N$, $\exists!$ symmetric SME with $n^* = N$; (ii) for generic $i \in [i^N, \bar{\imath}^C]$, $\exists!$ symmetric SME with $n^* \leq N$ ($<$ if $i > i^N$); (iii) for $i > \bar{\imath}^C$, $\nexists$ SME.*

Our environment satisfies all the properties under which Galenianos and Kircher (2012) demonstrate uniqueness of equilibrium when terms of trade of an indivisible good are determined by price posting with directed search. However, because of money, multiplicity is possible in our model when the buyer's payoff is zero and they randomize over entry decision. The difference is due to the finite-agent setup in Galenianos and Kircher (2012), which generates the better-reply security of Reny (1999). Given finite agents and zero payoff for buyers, sellers can always post a better terms of trade to increase trade probability and make buyer's surplus positive. This cannot happen in a model with infinitely many agents, since an individual seller has measure zero and cannot change his own trading probability by posting a different terms of trade.

To compare with Nash bargaining, the real DM price and the buyer's real balance under competitive search are always affected by i, and money is not superneutral, while it is still neutral. Therefore, the buyer's surplus from trade under competitive search adjusts endogenously with nominal interest rate, i.e., p^c is decreasing in i. As shown in Lagos and Rocheteau (2005) with divisible goods, higher anticipated inflation gives a larger share of surplus to buyers. Their results hold under indivisible goods. While under bargaining, the buyer's share of surplus is determined exogenously by the bargaining power, and does not adjust with nominal interest rate. If $i = 0$, holding money is costless. The DM price under competitive search then becomes

$$p^c = [1 - \varepsilon(n^*)]\, u + \varepsilon(n^*)\, c,$$

which is the same as the price under pure credit and bargaining with credit, given $\eta = \varepsilon(n)$.

Competitive search provides a natural environment to get (generically) unique equilibrium. Buyers direct their search to the sellers who give the highest expected payoff.

Competition among sellers guarantees that a buyer gets Ω from DM trade. The expected queue length adjusts continuously with the posted price, and the market-clearing price in the DM is uniquely determined when the expected queue length equals the buyer-seller ratio of the entire economy N. The fact that this adjustment mechanism does not exist under price posting and random search leads to the existence of a continuum of monetary equilibria, as in Green and Zhou (1998) and Jean et al. (2010).[5]

5 Asset

Now, instead of using money, buyers in the DM pay sellers with real assets. The total asset supply is fixed at A^s. Let φ_t be the CM price of real assets in terms of x_t, $A = A^s/N$ be the average amount of assets held by each buyer, and ρ be the dividend of real assets, which can be either positive or negative.

Buyers bring a into the CM and solve

$$W_t^b(a) = \Sigma + (\varphi_t + \rho)\, a + \max_{\hat{a}} \left\{ \beta V_{t+1}^b(\hat{a}) - \varphi_t \hat{a} \right\},$$

where $\hat{a}$ is the asset holding carried into the following DM. For a seller with a we have

$$W_t^s(a) = \Sigma + (\varphi_t + \rho)\, a + \max_{\hat{a}} \left\{ \beta V_{t+1}^s(\hat{a}) - \varphi_t \hat{a} \right\}. \tag{17}$$

The buyer's value function in the DM is

$$V_t^b(a) = \frac{\alpha(n)}{n} \left[u + W_t^b \left(a - \frac{p}{\varphi_t + \rho} \right) \right] + \left[1 - \frac{\alpha(n)}{n} \right] W_t^b(a), \tag{18}$$

where p is the price paid by the buyer for the DM good. Using $\partial W_t^b / \partial a = \varphi_t + \rho$, we can write

$$V_t^b(a) = \frac{\alpha(n)}{n}(u - p) + W_t^b(a) \tag{19}$$

$$V_t^s(a) = \alpha(n)(p - c) + W_t^s(a). \tag{20}$$

Sellers do not need assets for trading purposes in the DM, but they can still use assets as

[5] Apart from existence, our results differ quite substantially from those of Jean et al (2010). They consider price posting and random search to show a continuum of equilibria indexed by different real balances. Their result is driven by coordination failure from simultaneous moves by buyers and sellers. To obtain a unique equilibrium, they impose the assumptions of finite agents and sequential move.

a store of value. A necessary condition for sellers to hold assets is $\varphi_t = \beta(\varphi_{t+1} + \rho)$, i.e. the asset is priced at its fundamental value, where $\varphi_{t+1} + \rho$ is the real return on assets measured by CM goods.

5.1 Bargaining

The generalized Nash bargaining problem is the same as (7) with a different feasibility constraint $p \leq (\varphi_t + \rho)a$. Substitute V^b_{t+1} into W^b_t and we get the buyer's CM value function:

$$W^b_t(a) = \Sigma + (\varphi_t + \rho)a + \beta W^b_{t+1}(0) + \max_{\hat{a}} \left\{ \beta \frac{\alpha(n)}{n}(u - p) + \left[\beta\left(\varphi_{t+1} + \rho\right) - \varphi_t\right]\hat{a} \right\}.$$

Since $\beta(\varphi_{t+1} + \rho) \leq \varphi_t$ holds, the feasibility constraint binds and the bargaining solution implies $c \leq (\varphi_t + \rho)\hat{a} \leq (1 - \eta)u + \eta c$.

The buyer's problem can be rewritten as

$$\max_{\hat{a} \in [\underline{a}, \bar{a}]} \left\{ \beta \frac{\alpha(n)}{n} \left[u - \left(\varphi_{t+1} + \rho\right)\hat{a}\right] + \left[\beta\left(\varphi_{t+1} + \rho\right) - \varphi_t\right]\hat{a} \right\}, \tag{21}$$

where $\underline{a} = \frac{c}{\varphi_{t+1}+\rho}$ and $\bar{a} = \frac{(1-\eta)u+\eta c}{\varphi_{t+1}+\rho}$. The solution is $\hat{a}^*(\varphi_{t+1} + \rho) = c$. With bargaining, a buyer can again commit to not paying more than the seller's reservation price c. The buyer's value from participating in the DM is

$$v_n = \beta \left[\frac{\alpha(n)}{n} u + \left(1 - \frac{\alpha(n)}{n}\right) c\right] - \frac{\varphi_t c}{\varphi_{t+1} + \rho}. \tag{22}$$

The measure of DM buyers n^* is determined by the free entry condition, $v_{n^*} \geq 0$.

The asset prices in stationary equilibrium satisfy $\varphi_t = \varphi_{t+1}$. To establish equilibrium existence and uniqueness, we start by characterizing asset prices. There are two cases. When assets are held for store of value, they are priced fundamentally and both buyers and sellers hold them. When assets are held for liquidity purposes, only buyers hold assets.

Lemma 2 *Given n, the measure of active buyers in the DM: (i) for $\rho \geq (1 - \beta)cn/A^s$, $\varphi = \varphi^F = \rho/r$ and $\hat{a} \leq A$; (ii) for $\rho < (1 - \beta)cn/A^s$, $\varphi = cn/A^s - \rho > \varphi^F$ and $\hat{a} = A^s/n \geq A$.*

If the dividend of assets ρ is high, a buyer does not need to carry many assets for the

DM purchase, and the marginal holder of assets is a seller. Sellers only hold assets when they are at their fundamental price. In this case, the participation constraint (22) becomes $v_n(\varphi^F) = \beta(u-c)\alpha(n)/n$, which is positive for $n = N$. Hence, when $\rho \geq \rho^F = (1-\beta)cN/A^s$, the liquidity need of all buyers is satisfied and they all participate in the DM. The seller's asset holding is positive if $\rho > \rho^F$.

If ρ is low, the marginal holder of assets is a buyer, who cares about liquidity. The liquidity function drives up the asset price to be above its fundamental value, and sellers no longer hold assets. Substitute the asset prices into (22) and the buyer's participation constraint becomes

$$\frac{\alpha(n)}{n}(u-c) - \frac{r\varphi - \rho}{\varphi + \rho}c \geq 0. \tag{23}$$

Define the spread of assets $s = (r\varphi - \rho)/(\varphi + \rho)$. Notice that (23) is similar to (9), the buyer's participation constraint in the monetary economy. We can rewrite the spread as $1 + s = (1+r)\varphi/(\varphi + \rho)$ and $\varphi/(\varphi + \rho)$ is similar to ϕ_{t-1}/ϕ_t, with money. Hence, similar to the nominal interest rate i, s represents the cost of holding assets as the medium of exchange, and buyers never carry more real assets than the amount needed to pay for DM goods if $s > 0$.

The equilibrium measure of buyers in the DM n^* is determined by $f(n^*) = -\rho$. Define $\rho^N = -f(N)$, and all the buyers participate in the DM if $\rho \geq \rho^N$. Once $\rho < \rho^N$, buyers' participation starts to decrease, and eventually the DM will shut down. We summarize different cases of equilibria in the following proposition.

Proposition 5 *In the model with assets and bargaining: (i) for $\rho \geq \rho^F$, $\exists!$ SE with $\varphi = \varphi^F$ and $n^* = N$; (ii) for $\rho \in [\rho^N, \rho^F)$, $\exists!$ stable SE with $\varphi = \varphi^N > \varphi^F$ and $n^* = N$; (iii) for $\rho \in [\underline{\rho}, \rho^N)$, $\exists!$ stable SE with $\varphi = \varphi^{n^*} > \varphi^F$ and $n^* < N$; (iv) for $\rho < \underline{\rho}$, $\nexists$ equilibrium with an active DM.*

Corollary 1 *For $\rho \in (\underline{\rho}, 0)$, $\exists!$ unstable equilibrium with $\varphi = \varphi^{n^*} > \varphi^F$ and $n^* < N$.*

In Figure 1, while the dashed curves below the horizon represent the buyer's participation constraint $f(n)$, the solid curves above the horizontal axis show how the asset price φ changes with respect to the dividend ρ. When ρ is large enough, assets are priced at the fundamental value and they are not affected by the buyer's participation in the DM. For any dividend between ρ^N and ρ^F, the asset price is above its fundamental price and decreasing in ρ. When ρ becomes small enough, i.e. $\rho < \rho^N$, both dividend and the buyers' participation n^* affect the asset price in opposite directions, and φ is increasing in ρ in stable equilibrium. For $\rho < 0$, the asset prices are still positive due to the liquidity premium in facilitating DM trade, and the two prices correspond to two stationary equilibria.

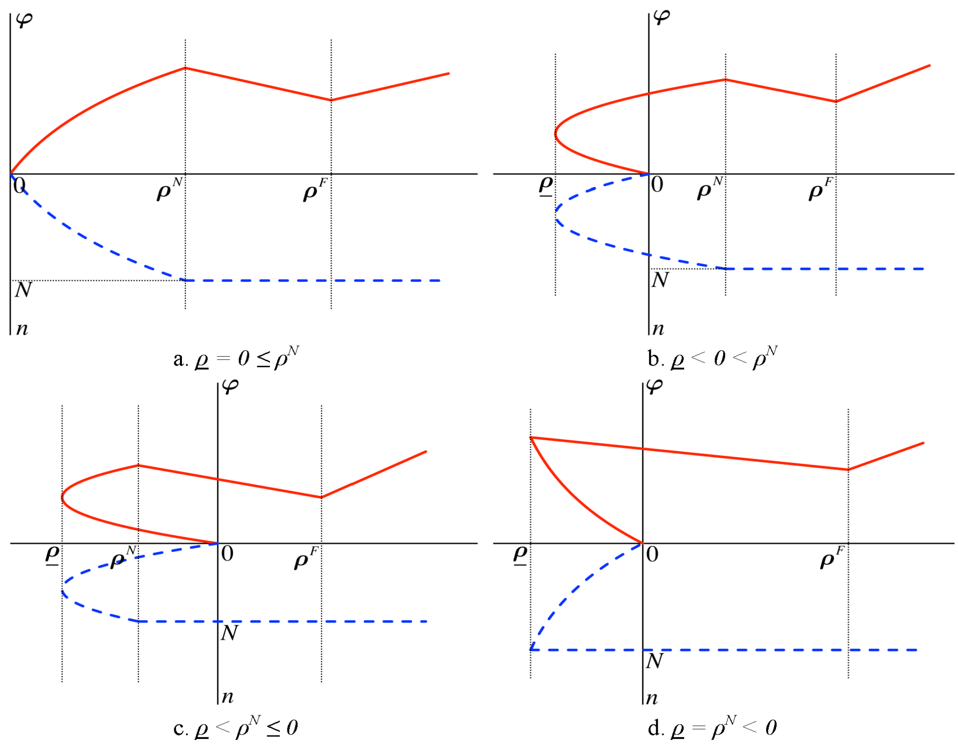

Figure 1: Asset with Bargaining

There are two different levels of buyer's participation in the DM, high and low. When the equilibrium participation is high, a larger liquidity demand drives up the asset price φ, which implies a smaller asset spread s, since $\partial s/\partial\varphi < 0$ when $\rho < 0$. Buyers now face a low probability of trade in the DM, but they are compensated by a small cost of holding assets. Similarly, when the equilibrium participation is low, buyers receive a high probability to trade with a large cost of holding assets. When $\rho \geq 0$, the coordination problem does not exist, since $\partial s/\partial\varphi \geq 0$, and hence equilibrium is unique. Unlike assets, the cost of holding credit or money is exogenously given at zero or i, and it does not depend on the buyer's participation in the DM. Therefore, this coordination effect does not exist in the credit or monetary economy and there is always a unique equilibrium.

When $\rho < 0$, the negative dividend is similar to a storage cost, akin to Kiyotaki and Wright (1989). When $\rho = 0$, assets are equivalent to money with a constant supply. Unlike money, the cost of holding assets depends on dividend and the buyer's participation, while i is an exogenous policy choice leading to a unique equilibrium. In the monetary economy, a higher participation in the DM implies lower probability of trade for buyers. Since the cost of holding money is not adjusting with n, buyers are strictly worse off, and the coordination problem no longer exists. For $\rho > 0$, an equilibrium with assets always exists. However, monetary equilibrium may not exist with deflation, if the surplus from trade $(u - c)/c$ is small enough, since the cost of holding money is independent of n. When $s = 0$, carrying assets becomes costless, the asset economy is comparable to the credit economy in Section 3.1.

5.2 Competitive Search

Similar to (13), the seller's price posting problem after substituting p from the constraint yields

$$\max_n \pi(n) = \alpha(n) \left[\frac{\alpha(n) u - n\Omega}{\alpha(n) + ns} - c \right]. \tag{24}$$

In equilibrium, the optimal queue length is consistent with free entry

$$\frac{\alpha(n^*)}{n^*}(u - p^c) - sp^c = \Omega \geq 0, \tag{25}$$

and p^c is the seller's optimal price

$$p^c = \frac{\alpha(n^*) \{[1 - \varepsilon(n^*)] u + \varepsilon(n^*)c\} + \varepsilon(n^*)n^* sc}{\alpha(n^*) + \varepsilon(n^*)n^* s}.$$

We study the existence and uniqueness of equilibrium by equating the aggregate demand and supply of liquidity. The aggregate demand of liquidity $L^d = n^* p^c$ is a function of the spread s. Given a one-to-one mapping from the asset price φ to s, the aggregate supply $L^s = (\varphi + \rho)A^s$ is also a function of s. The aggregate demand and supply of liquidity are characterized by the following lemmas.

Lemma 3 *There exist $\bar{s}^C \geq r$ and $s^N \leq \bar{s}^C$, such that: (i) for $s < s^N$, $\exists!$ L^d with $n^* = N$, and $dL^d/ds < 0$; (ii) for generic $s \in [s^N, \bar{s}^C]$, $\exists!$ L^d with $n^* \leq N$ ($<$ if $s > s^N$), and $dL^d/ds < 0$; (iii) for $s > \bar{s}^C$, $\nexists$ $n^* > 0$ and L^d is not well-defined.*

Recall $s = (r\varphi - \rho)/(\varphi + \rho)$ is the spread of assets and $\partial s/\partial \rho < 0$. As shown in Lemma 3, if the asset dividend is low enough and the cost of holding asset is high enough, the DM will shut down. As long as the DM operates and L^d is well-defined, it is monotonically decreasing in s. The DM participation of buyers varies depending on different values of ρ hence s. Next lemma characterizes the aggregate supply of liquidity.

Lemma 4 *For $\rho < 0$, L^s is convex and $dL^s/ds < 0$; for $\rho = 0$, L^s is perfectly elastic at $s = r$; for $\rho \in (0, \rho^F)$, L^s is concave and $dL^s/ds > 0$; for $\rho \geq \rho^F$, L^s is perfectly elastic at $s = 0$.*

Notice that the spread of assets can be rewritten in two parts, $s = r-(1+r)\rho/(\varphi+\rho)$. If $\rho = 0$ and assets have no dividend return, the second term vanishes and only the discount factor is left. In the following, we first determine s^* by $L^d(s) = L^s(s)$, and then back out asset price and participation in equilibrium.

Proposition 6 *In the model with assets and competitive search, there exist ρ^F, ρ^N, and $\underline{\rho}$, such that: (i) for $\rho \geq \rho^F$, $\exists!$ symmetric SE with $\varphi = \varphi^F$ and $n^* = N$; (ii) for $\rho \in (\rho^N, \rho^F)$, $\exists!$ symmetric SE with $\varphi = \varphi^N > \varphi^F$ and $n^* = N$; (iii) for (generic) $\rho \in [\underline{\rho}, \rho^N]$, $\exists!$ symmetric SE if $\rho > 0$ ($\rho \leq 0$), with $\varphi = \varphi^{n^*} > \varphi^F$ and $n^* \leq N$ ($<$ if $\rho < \rho^N$); (iv) for $\rho < \underline{\rho}$, $\nexists$ equilibrium with an active DM.*

Figure 2 shows the relationship between equilibrium participation n^* and dividend ρ by the dashed curves below the horizontal axis. Above the horizon, the solid curves represent asset price φ as a function of ρ. As long as the dividend is high enough, all buyers participate in the DM and assets are priced at the fundamental value. If ρ is smaller than ρ^N, not all buyers enter the DM. Since a larger dividend implies a smaller spread s, i.e. a lower cost of holding assets, the buyers' participation is monotonically increasing in ρ. However, the asset price φ may change in a non-monotonic way with respect to ρ. Equating the demand and supply of liquidity, we get the asset price $\varphi = L^d/A^s - \rho$, which is the difference between the return of holding the asset and its dividend. As ρ gets larger, the asset return also increases due to a higher demand induced by ρ. Then, the change in asset price really depends on how much the liquidity demand responds to ρ, which is undetermined under general parameter values.

Similar to bargaining, multiple equilibria are possible only with $\rho \leq 0$. With bargaining, multiplicity is supported by a continuum of ρ. This is because buyers search randomly and the equilibrium price in the bargaining game is the seller's reservation value, independent of the market tightness in the DM. With competitive search, the prices posted

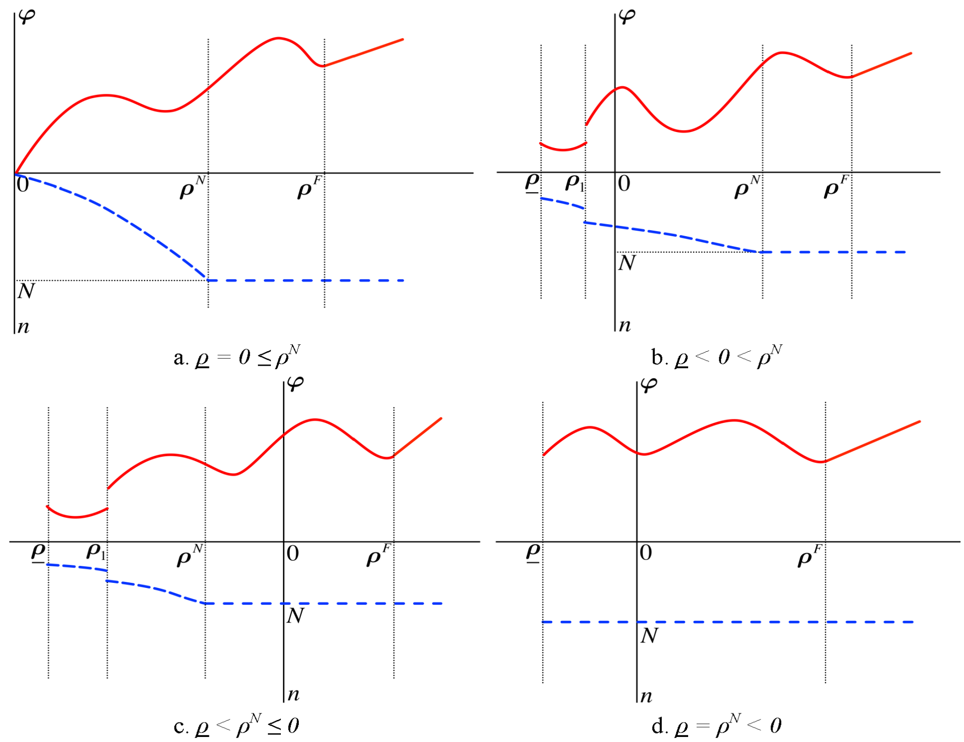

Figure 2: Asset with Competitive Search

by sellers direct the buyers' search behavior and serve as a coordination device. As a result, the set of ρ supporting multiple equilibrium is countable with measure zero, and equilibrium is generically unique.

Proposition 6 shows uniqueness for $0 < \rho < \rho^N$, i.e. $r > s > s^N$. With money, there may still exist multiple equilibria for $r > i > i^N$. This is due to the cost of holding assets being endogenously determined while the cost of holding money is an exogenous policy variable. For $i > i^N$, the liquidity demand for money may not be unique for a countable number of interest rates. For these exogenous i, there are multiple equilibria featuring different real money balances. With assets, the liquidity demand can have multiple values at a countable number of s as well, but the spread is endogenously determined by $L^d = L^s$. According to Lemma 4, L^s is monotonically increasing in s, and there is a unique asset spread given n. For $\rho \geq 0$, the asset spread is increasing in n. With more buyers entering the DM, they face a higher cost of holding assets and a lower probability of trade. There is no coordination problem and a unique equilibrium n^* with a unique asset spread exists. We also obtain uniqueness if the cost of holding assets is a constant and independent of n, such as $s = r$ with $\rho = 0$. When holding assets is costless, i.e. $s = 0$, and the asset is priced at fundamental value, the equilibrium has the same price and participation in the DM as the equilibrium with pure credit.

5.3 Discussion

Indivisibility matters mainly from losing an intensive margin of adjustment. It makes the available surplus fixed without endogenous participation of buyers in the DM. In addition, different pricing mechanisms yield different results, and it matters if pure credit is used compared to money or assets.

To summarize this, we catalog the different cases. Let $n \leq N$ be the active measure of buyers in the DM. Let $B_L^j(n)$ be buyers' benefit from participation, $L \in \{c, m, a\}$ be the three types of liquidity, credit, money, or assets, and $j \in \{b, c\}$ the type of pricing mechanisms, bargaining or competitive search. Let p^j be the equilibrium price under the mechanism j.

In the credit economy, we find that buyers participate in the DM if

$$\begin{aligned} B_c^b(n) &= \left(u - p^b\right) \frac{\alpha(n)}{n} \geq 0 \\ B_c^c(n) &= (u - p^c) \frac{\alpha(n)}{n} \geq 0. \end{aligned}$$

The main difference is the bargained price being independent of n, but not under competitive search. As long as $B_c^i(N) > 0$, all potential buyers participate in the DM.

In the monetary economy, we find

$$\begin{aligned} B_m^b(n) &= (u - p^b) \frac{\alpha(n)}{n} \geq ip^b \\ B_m^c(n) &= (u - p^c) \frac{\alpha(n)}{n} \geq ip^c. \end{aligned}$$

Since $\alpha(n)/n$ is decreasing in n, under bargaining, for large enough i, $B_m^b(N) < ip^b$ and not all buyers participate in the DM. With competitive search, p^c is increasing in n and decreasing in i. Higher i reduces p^c, which increases $B_m^c(n)$, $\forall n$. But it also increases ip^c. This generates the potential for multiple equilibria with $n < N$. However, as we show for generic values of i, these possibilities are measure zero. Thus, monetary equilibrium is generically unique.

For the asset economy, we find

$$\begin{aligned} B_a^b(n) &= (u - p^b)\frac{\alpha(n)}{n} \geq sp^b \\ B_a^c(n) &= (u - p^c)\frac{\alpha(n)}{n} \geq sp^c, \end{aligned}$$

where the spread $s(n, \rho)$ is decreasing in n and increasing in ρ. With bargaining, we find unique n when $\rho > 0$, but when $\rho < 0$, there are two equilibrium n for a range of ρ. With competitive search, the asset equilibrium is generically unique even when $\rho \leq 0$. The price reacts to the nominal interest rate, dividend value, and endogenous participation.

6 Lotteries

In an environment with indivisible goods, one can consider lotteries. To do so, we reconsider the three liquidity possibilities and two mechanisms as above. Let $E = \mathcal{P} \times \{0, 1\}$ denote the space of trading events, and $\mathcal{E}$ the Borel σ-algebra. Define a lottery to be a probability measure ω on the measurable space $(E, \mathcal{E})$. We can write $\omega(p, q) = \omega_q(q)\omega_{p|q}(p)$ where $\omega_q(q)$ is the marginal probability measure of q and $\omega_{p|q}(p)$ is the conditional probability measure of p on q. Without loss of generality, as shown in Berentsen et al. (2002), we restrict attention to $\tau = \Pr\{q = 1\}$ and $1 - \tau = \Pr\{q = 0\}$, and $\omega_{p|0}(p) = \omega_{p|1}(p) = 1$. Randomization is only useful on q because Q is non-convex. Thus, $\tau \in [0, 1]$ is the probability that the good is produced and traded.

6.1 Credit with Lotteries

The buyer's payoff in the DM is

$$V_t^b = \frac{\alpha(N)}{N}\left[\tau u + W_t^b(p)\right] + \left[1 - \frac{\alpha(N)}{N}\right] W_t^b(0). \tag{26}$$

If a buyer gets to trade, he gets credit p, to be paid in the next CM, and gets τu from consumption in the DM. Payoffs for buyers and sellers in the DM can be rewritten as

$$\begin{aligned} V_t^b &= W_t^b(0) + \frac{\alpha(N)}{N}(\tau u - p) \\ V_t^s &= W_t^s(0) + \alpha(N)(p - \tau c). \end{aligned}$$

First, consider bargaining,

$$\max_{p,\tau} (\tau u - p)^{\eta} (p - \tau c)^{1-\eta} \text{ s.t. } p \leq D,\, \tau \leq 1.$$

Proposition 7 *In the credit model with bargaining and lotteries, ∃! SE if $D \geq c$, characterized by*

$$\left(p^B, \tau^B\right) = \begin{cases} (\bar{p}, 1) & \text{if } D > \bar{p}^B \\ (D, 1) & \text{if } \underline{p}^B \leq D \leq \bar{p}^B \\ \left(D, D/\underline{p}\right) & \text{if } D < \underline{p}^B \end{cases}$$

where $\bar{p}^B = (1-\eta)u + \eta c$ and $\underline{p}^B = uc/(\eta u + (1-\eta)c)$.

The expected total surplus from trade is $\tau(u - c)$. As long as the credit constraint is not too tight, i.e. $D \geq \underline{p}^B$, the surplus is maximized at $\tau^B = 1$. If the credit constraint is too tight, they can adjust/lower τ to compensate the seller, while they could not do that without lotteries. The solution without lotteries is a subset of the one above forcing $\tau = 1$. Note that all buyers are always active in the DM with credit, since $\alpha(n)\left(\tau^B u - p^B\right)/n > 0$ for all $n \leq N$.

Next, consider competitive search. To attract queue length n, sellers must post price p to guarantee buyers an expected utility of Ω. Solve for p from the buyer's participation constraint, and substitute into the seller's objective function:

$$\max_{\tau,n} \alpha(n) \left[\tau (u - c) - \frac{n\Omega}{\alpha(n)}\right] \text{ s.t. } \tau u - \frac{n\Omega}{\alpha(n)} \leq D,\, \tau \leq 1.$$

Proposition 8 *In the credit model with competitive search and lotteries, ∃! symmetric SE if $D \geq c$, characterized by*

$$\left(p^C, \tau^C\right) = \begin{cases} \bar{p}^C, 1 & \text{if } D > \bar{p}^C \\ D, 1 & \text{if } \underline{p}^C \leq D \leq \bar{p}^C \\ D, D/\underline{d}^C & \text{if } D < \underline{p}^C \end{cases}$$

where $\bar{p}^C = (1-\varepsilon(n))\,u + \varepsilon(n)\,c$, $\underline{p}^C = uc/(\varepsilon(n)u + (1-\varepsilon(n))c)$, and $n = N$.

It is easy to check $\alpha(N)\left(\tau^C u - p^C\right)/N > 0$. When $\varepsilon(N) = \eta$, the equilibrium price is identical to the one with bargaining.

6.2 Money with Lotteries

We consider bargaining first,

$$\max_{p,\tau} (\tau u - p)^{\eta} (p - \tau c)^{1-\eta} \text{ s.t. } p \leq \phi m,\ \tau \leq 1, \text{ and } \tau u \geq p \geq \tau c.$$

Lemma 5 *The solution to the bargaining problem is*

$$\left(p^B, \tau^B\right) = \begin{cases} \left(\bar{p}^B, 1\right) & \text{if } \phi m > \bar{p}^B \\ (\phi m, 1) & \text{if } \underline{p}^B \leq \phi m \leq \bar{p}^B \\ \left(\phi m, \phi m / \underline{p}^B\right) & \text{if } c \leq \phi m < \underline{p}^B \\ (0, 0) & \text{if } \phi m < c \end{cases}$$

where $\bar{p}^B = (1-\eta)\, u + \eta c$ *and* $\underline{p}^B = uc/(\eta u + (1-\eta)c)$.

Buyer's payoff in the CM is

$$W_t^b(m) = \Sigma + \phi_t (m + T) + \beta W_{t+1}^b (0) + \beta \max_{\hat{m}} v(\hat{m}),$$

where $v(\hat{m}) = \alpha(n)(\tau^B u - p^B)/n - i\phi_{t+1}\hat{m}$.

Proposition 9 *In the monetary model with bargaining and lotteries: (i) For* $i \leq i^N$, $\exists!$ *SME with* $\phi_{t+1}\hat{m} = \underline{p}^B$, $\tau^B = 1$ *and* $n^* = N$*; (ii) for* $i \in (i^N, \bar{\imath}^B)$, $\exists!$ *SME with* $\phi_{t+1}\hat{m} = \underline{p}^B$, $\tau^B = 1$ *and* $n^* < N$*; (iii) for* $i \geq \bar{\imath}^B$, $\nexists$ *SME.*

Notice first $\phi_{t+1}\hat{m} = \underline{p}^B$ and the measure of participating buyers do not decrease with inflation when $i \leq i^N$. Money is still superneutral. For $i \in (i^N, \bar{\imath}^B)$, real balances stay constant but n^* changes with i. Second, lotteries benefit sellers. With lotteries, the seller's surplus from DM trade is $\underline{p}^B - c$, compared to zero surplus from trade without bargaining over lotteries. Because of lotteries, buyers now bring exactly enough money to achieve the maximum expected surplus from trade at $\tau^B = 1$. Third, introducing lotteries makes it harder for a monetary equilibrium to exist.[6] Fourth, with lotteries, the two cutoff values of nominal interest rate increase with the buyer's bargaining power η. Finally, compared to Berentsen et al. (2002), the probability τ^B does not change with respect to the buyer's bargaining power or the inflation rate. Introducing lotteries with

[6] Allowing buyers and sellers to bargain over lotteries in an existing monetary equilibrium without lotteries will cause the equilibrium to collapse. This can be seen from the fact that $\bar{\imath}^B = \eta(u-c)/c$ in the lottery case is smaller than $\bar{\imath}^B = (u-c)/c$ in the case without lotteries.

indivisible goods and *divisible money*, the total surplus from trade is affected but not price. However, introducing lotteries with *indivisible money* and *divisible goods*, the total surplus from trade stays the same but price changes according to the value of lotteries in equilibrium.

Under competitive search, seller's price posting problem is

$$\max_{p,\tau,n} \alpha(n)(p - \tau c) \text{ s.t. } \frac{\alpha(n)}{n}(\tau u - p) - ip = \Omega,\ \tau \leq 1.$$

Proposition 10 *In the monetary model with competitive search and lotteries: (i) For* $i < i^N$, $\exists!$ *symmetric SME with* $\phi_{t+1}\hat{m} = p^c$, $\tau^C = 1$ *and* $n^* = N$*; (ii) for generic* $i \in [i^N, \bar{\imath}^C]$, $\exists!$ *symmetric SME with* $\phi_{t+1}\hat{m} = p^c$, $\tau^C = 1$ *and* $n^* \leq N$*; (iii) for* $i > \bar{\imath}^C$, $\nexists$ *SME.*

In a monetary equilibrium with competitive search, lotteries are never used by sellers no matter how many buyers participate in the DM. Sellers have the opportunity to post prices to get the highest possible profit, choosing to guarantee $\tau^C = 1$. This is similar to Guerrieri, Shimer and Wright (2010) with divisible goods and money, competitive search and lotteries under adverse selection.[7] Sellers are able to maximize their expected profits without lotteries. Hence with lotteries, equilibrium prices and the cutoff value of the nominal interest rate are the same as before. Things are different with bargaining. With bargaining lotteries matter, i.e. $\underline{p}^B > c$.

6.3 Asset with Lotteries

Finally, we introduce lotteries with assets as the medium of exchange. The generalized Nash bargaining problem is

$$\max_{p,\tau} (\tau u - p)^\eta (p - \tau c)^{1-\eta} \text{ s.t. } p \leq (\varphi + \rho) a,\ \tau \leq 1,$$

with $\tau u \geq p$ and $p \geq \tau c$.

[7] With indivisible money and divisble goods, Julien, Kennes and King (2008) show that lotteries matter with competitive search.

Lemma 6 *The solution to the bargaining problem is*

$$\left(p^B, \tau^B\right) = \begin{cases} \left(\bar{p}^B, 1\right) & \text{if } (\varphi+\rho)a > \bar{p}^B \\ ((\varphi+\rho)a, 1) & \text{if } \underline{p}^B \le (\varphi+\rho)a \le \bar{p}^B \\ \left((\varphi+\rho)a, (\varphi+\rho)a/\underline{p}^B\right) & \text{if } c \le (\varphi+\rho)a < \underline{p}^B \\ (0,0) & \text{if } (\varphi+\rho)a < c \end{cases}$$

where $\bar{p}^B = (1-\eta)\,u + \eta c$ *and* $\underline{p}^B = uc/(\eta u + (1-\eta)c)$*.*

Buyer's CM value is

$$W_t^b(a) = \Sigma + (\varphi_t + \rho)a + \beta W_{t+1}^b(0) + \beta \max_{\hat{a}} v(\hat{a}),$$

where $v(\hat{a}) = \alpha(n)(\tau^B u - p^B)/n - s(\varphi_{t+1} + \rho)\hat{a}$.

Proposition 11 *In the asset model with bargaining and lotteries: (i) for* $\rho \ge \rho^F$*,* $\exists!$ *SE with* $\varphi = \varphi^F$ *and* $n^* = N$*; (ii) for* $\rho \in [\rho^N, \rho^F)$*,* $\exists!$ *stable SE with* $\varphi = \varphi^N > \varphi^F$ *and* $n^* = N$*; (iii) for* $\rho \in [\underline{\rho}, \rho^N)$*,* $\exists!$ *stable SE with* $\varphi = \varphi^{n^*} > \varphi^F$ *and* $n^* < N$*; (iv) for* $\rho \in (\underline{\rho}, 0)$*,* $\exists!$ *unstable SE; (v) for* $\rho < \underline{\rho}$*,* $\nexists$ *equilibrium with an active DM; (vi)* $p^B = \underline{p}^B$ *and* $\tau^B = 1$ *hold for (i)-(iii).*

The results are very similar to the case with money. Lotteries are not used in equilibrium. $\underline{p}^B$ and τ^B do not change with ρ. Compared to the case of bargaining with assets and no lotteries, we still get a continuum of equilibria for $\rho \in (\underline{\rho}, 0)$, since the coordination problem still exists.

Finally, with competitive search we have,

Proposition 12 *In the asset model with competitive search and lotteries, there exist* ρ^F*,* ρ^N*, and* $\underline{\rho}$*, such that: (i) for* $\rho \ge \rho^F$*,* $\exists!$ *symmetric SE with* $\varphi = \varphi^F$ *and* $n^* = N$*; (ii) for* $\rho \in (\rho^N, \rho^F)$*,* $\exists!$ *symmetric SE with* $\varphi = \varphi^N > \varphi^F$ *and* $n^* = N$*; (iii) for (generic)* $\rho \in [\underline{\rho}, \rho^N]$*,* $\exists!$ *symmetric SE if* $\rho > 0$ *(*$\rho \le 0$*), with* $\varphi = \varphi^{n^*} > \varphi^F$ *and* $n^* \le N$*; (iv) for* $\rho < \underline{\rho}$*,* $\nexists$ *equilibrium with an active DM; (v)* $\tau^C = 1$ *holds for (i)-(iii).*

7 Conclusion

In this paper, we use a general equilibrium model to study the trade of indivisible goods in frictional markets. Indivisibility matters, especially when terms of trade are determined by bargaining with money or assets. The bargained price gives sellers no surplus and is independent of the nominal interest rate or the dividend on assets. Money is then superneutral as long as all buyers participate in the market. Introducing lotteries does not change this. Under competitive search, the price that depends on the nominal interest rate with money, the dividend with assets, and the number of buyers in the market. Lotteries do not matter under competitive search, but do under bargaining.

In the pure credit economy, we show uniqueness under bargaining and competitive search. We also show uniqueness under bargaining in the monetary economy. Under competitive search, we get uniqueness for generic parameters. In the asset economy, under bargaining, the equilibrium is unique as long as the asset dividend is non-negative. With a negative dividend we find two equilibria, with low and high participation. The congestion nature of the matching technology, generates a concave net benefit for buyers in the number of active buyers. This leads to a coordination problem and two equilibria. With competitive search and price posting, we find a unique equilibrium for positive dividends. With zero and negative dividends, we find the equilibrium to be generically unique. Using price posting as a coordination device solves the problem present under bargaining.

Overall, the consequences of indivisibility on the goods side matter and differ from indivisibility on the money side. Lotteries cannot recover conventional results. In particular, lotteries cannot reestablish the link between real balances and anticipated inflation under bargaining. Indivisibility may also affect the bargaining outcome because it isolates the good's price from the nominal interest rate, the dividend value, and the number of buyers. Price posting with competitive search reestablishes the link and generically produces a unique equilibrium. While we have focused on stationary equilibrium, the model can easily be used to study asset price dynamics. We leave this for future research.

Appendix

Proof of Proposition 1. The stationary equilibrium with credit is characterized by the solution to the bargaining problem. Using λ as the multiplier on the credit constraint yields the following Kuhn-Tucker conditions:

$$\begin{aligned} 0 &= (1-\eta)(u-p)^{\eta}(p-c)^{-\eta} - \eta(u-p)^{\eta-1}(p-c)^{1-\eta} - \lambda \\ 0 &= \lambda\,(D-p)\,. \end{aligned}$$

If $\lambda = 0$, then $p = (1-\eta)u + \eta c \equiv \bar{p}^B$. However, if $\lambda > 0$, then $p = D$. Finally, we need $D \geq c$ to guarantee non-negative surplus for sellers. ■

Proof of Proposition 2. The proof is similar to Proposition 1. ■

Proof of Proposition 3. First, $i \leq \bar{\imath}^N = \Psi(N)$ implies $v_N(c) \geq v_N(0)$ and hence (i). For (ii), we need $\lim_{n\to 0}\Psi(n) = (u-c)/c = \bar{\imath}^B$, which is assured by the assumptions of $\alpha(n)$. Finally, for $i \geq \bar{\imath}^B$, $v_n(c) < v_n(0)$ for all $n > 0$, and the DM is inactive. ■

Proof of Lemma 1. To prove the existence and uniqueness of L^d, it is sufficient to show the existence and uniqueness of $n^* > 0$. Substitute p^c into (15) and we get $\alpha\varepsilon\,(u-c)\,i + \alpha^2\varepsilon\,(u-c)\,/n^* \;\geq\; \alpha\,[(1-\varepsilon)\,u + \varepsilon c]\,i + \varepsilon n^* c i^2$. Define $h(n^*, i) = \alpha\varepsilon(u-c)i + \alpha^2\varepsilon(u-c)/n^* - \alpha[(1-\varepsilon)u + \varepsilon c]i - \varepsilon n^* c i^2$. Given any $n \in (0, N]$, $h(n, i) = 0$ is a quadratic function in i, with two real solutions of opposite signs. The positive solution $i_+(n)$, satisfying $h(n, i_+) = 0$, is an implicit function of n. Let $i_+(0) = \lim_{n\to 0} i_+(n) < \infty$. It is easy to show $i_+(n)$ is continuous on $[0, N]$. Define i^N by $h(N, i^N) = 0$ and $\bar{\imath}^C = \max_{n\in[0,N]} i_+(n)$. For $i < i^N$, $h(N, i) > 0$ then $n^* = N$. Thus, $L^d = Np^c(N, i)$ is unique, and $\partial L^d/\partial i = N\partial p^c(N, i)/\partial i < 0$, hence (i). For $i > \bar{\imath}^C$, $h(n^*, i) < 0\ \forall n^*$, and the free-entry condition does not hold since $\alpha(n^*)(u-p^c)/n^* - ip^c < 0$, hence (iii).

Regarding (ii), for $i \leq \bar{\imath}^C$, $h(n^*, i) = 0$ always holds for some $n^* > 0$, and L^d exists. To show that L^d is generically unique and monotone, consider $L^d = n^* p^c$ and $dL^d/di = \partial L^d/\partial i + (\partial L^d/\partial n^*)(\partial n^*/\partial i)$. Given $h(n^*, i) = 0$, $L^d = \alpha(n^*)n^* u/[\alpha(n^*) + in^*]$, hence $\partial L^d/\partial i < 0$ and $\partial L^d/\partial n^* > 0$. Then, it is sufficient to show n^* is generically unique and $\partial n^*/\partial i < 0$. Next, we want to show that n^* is unique and $\partial n^*/\partial i < 0$ for generic i. Suppose $\pi(n_1^*, i) = \pi(n_2^*, i) = \max_n \pi(n, i)$ and $n_2^* > n_1^*$. Then, n_1^* is the minimum n maximizing $\pi(n, i)$, and $\pi(n_1^*, i) > \pi(n, i)$, $\forall n < n_1^*$. For small $\varepsilon > 0$, $\pi(n_1^*, i+\varepsilon) > \pi(n, i+\varepsilon)$ also holds for $n < n_1^*$ due to continuity. If $\partial^2\pi/\partial i\partial n^* < 0$, then $\pi(n_1^*, i+\varepsilon) > \pi(n_2^*, i+\varepsilon)$, and there is a unique global maximizer in the neighborhood of n_1^*. Finally, we need to show $\partial^2\pi/\partial i\partial n^* < 0$. Derive $\partial\pi/\partial n$ from (14),

$$\frac{\partial\pi}{\partial n} = \frac{(\alpha + in)\left[(u-c)\,\alpha' - ic\right] - i\,(1-\varepsilon)\left[(u-c)\,\alpha - inc\right]}{(\alpha + in)^2/\alpha}.$$

Define $T(i) = (\alpha+in)[(u-c)\alpha' - ic] - i(1-\varepsilon)[(u-c)\alpha - inc]$, and $T'(i) = n[(u-c)\alpha' - ic] - (\alpha+in)c - (1-\varepsilon)[(u-c)\alpha - inc] + inc(1-\varepsilon)$. Since $T_{n=n^*} = 0$, $\partial^2\pi/\partial i\partial n^* = T'(i)/[(\alpha+in^*)^2/\alpha]$. With $\alpha(u-c) - in^*c > 0$, we have

$$T'_{n=n^*}(i) = \frac{-\left[\alpha\,(u-c) - in^*c\right](1-\varepsilon)\,\alpha - c\,(\alpha+in^*)\,(\alpha+in^*\varepsilon)}{\alpha + in^*} < 0,$$

and $\partial^2\pi/\partial i\partial n^* < 0$ holds. Although $\arg\max_n \pi(n,i)$ may have more than one solution for some $i \geq i^{NC}$, the set of such i has measure zero, hence (ii). ■

Proof of Proposition 4. First, for $i > \bar{\imath}^C$, $n^* < 0$ as shown in Lemma 1, and there is no monetary equilibrium, hence (iii). For $i < i^N$, L^d is unique and monotonically decreasing in i. Hence, given i, there exists a unique real money holding $\phi_{t+1}\hat{m} = p^c$ and a unique SME. Since $h(N,i) > 0$ and $\alpha(N)(u-p^c)/N - ip^c > 0$, we have $n^* = N$, thus (i). Finally, as shown in the proof of Lemma 1, L^d is generically unique and $\partial L^d/\partial i < 0$ for $i \in [i^N, \bar{\imath}^C]$. Therefore, there exists a generically unique real balance $\phi_{t+1}\hat{m}$ and symmetric SME with $n^* \leq N$. The inequality is strict if $i > i^N$. ■

Proof of Lemma 2. If assets are priced fundamentally, then $\varphi = \varphi^F = \rho/r$. If $\varphi > \varphi^F$, buyers with measure n hold the assets and $\hat{a} = A^s/n \geq A^s/N = A$. From (21), the individual buyer's demand for assets is $\hat{a}^* = c/(\varphi+\rho)$. Equating demand with supply yields $\varphi = cn/A^s - \rho$. We need $\rho < (1-\beta)cn/A^s$ to guarantee $cn/A^s - \rho > \varphi^F$. Finally, when $\varphi = \varphi^F$, $\hat{a} = (1-\beta)c/\rho \leq A$, and sellers also hold assets. ■

Proof of Proposition 5. (i) is straightforward from Lemma (2). To establish uniqueness, notice that f is continuous, satisfying $f(0) = 0$ and $f''(n) < 0$. In Figure 1, $f(n)$ is represented by the dashed curves below horizontal axis on $[0,N]$. For $\rho \geq \rho^F > 0$, there is a unique n^* satisfying $f(n^*) = -\rho$. Define $\underline{\rho} = -\max_{n\in[0,N]} f(n)$. For $\rho < \underline{\rho} \leq 0$, $f(n) < -\rho \ \forall n > 0$, and the DM shuts down, hence (iv).

Regarding (ii) and (iii), first consider the case $\underline{\rho} = 0$ in Figure 1a. Since f is decreasing in n on $[\underline{\rho}, \rho^F)$, there exists a unique equilibrium indexed by n^*. For $\rho \in [\rho^N, \rho^F)$, all buyers participate in the DM. Then, the asset price is $\varphi^N = cN/A^s - \rho > \varphi^F$. For $\rho \in [\underline{\rho}, \rho^N)$, $f(N) < -\rho$ and $n^* < N$. It is easy to check $\varphi^{n^*} = cn^*/A^s - \rho > \varphi^F$ using $\rho = -f(n^*)$. Next, consider $\underline{\rho} < 0$. For $\rho \in [0, \rho^F)$, as in Figure 1b, $f(n) = -\rho$ has a unique solution

$n^* > 0$, and the above results hold. Then, Figure 1c and 1d show that, for $\rho \in (\underline{\rho}, 0)$, $f(n) = -\rho$ has exactly two solutions, denoted as $n_1^* > n_2^* > 0$. One can easily prove $n_1^* = N$ for $\rho \geq \rho^N$ and $n_1^* < N$ otherwise. Finally, to show the equilibrium at n_2^* is unstable, notice that for $n < n_2^*$, $f(n) < -\rho$, and buyers want to exit the DM until $n = 0$; for $n > n_2^*$, buyers want to enter the DM until $n = n_1^*$. ■

Proof of Lemma 3. We only need to show $\bar{s}^C \geq r$, and the rest of the proof is similar to Lemma 1, with the cost of holding assets being s instead of i. We prove $\bar{s}^C \geq r$ by contradiction. Suppose $\bar{s}^C < r$, then for $s_1 = (r\varphi_1 - \rho_1)/(\varphi_1 + \rho_1) \in (\bar{s}^C, r)$, $\rho_1 > 0$ and $n_1^* = 0$. Hence, $\varphi_1 = \varphi_1^F$ and $s_1 = 0$, contradicting $s_1 > \bar{s}^C > 0$. ■

Proof of Lemma 4. If assets are priced at the fundamental value, then all buyers participate in the DM and $s = 0$. Let $\rho^F = (1-\beta)p^C_{N,s=0}/A$. If $\rho \geq \rho^F$, the average asset holding $(\varphi + \rho)A^s/n \geq (\varphi^F + \rho)A^s/n \geq \rho^F A/(1-\beta) = p^C_{N,s=0}$. The liquidity need for assets is satisfied and the marginal holders of assets only care about the store of value function. Hence, $\varphi = \varphi^F$ and $s = 0$. If $\rho = 0$, the cost of holding assets is $s = r$. If $\rho < \rho^F$ and $\rho \neq 0$, substitute s into the liquidity supply and $L^s = (1+r)\rho A^s/(r-s)$, with $\partial L^s/\partial s = (1+r)\rho A^s/(r-s)^2$ and $\partial^2 L^s/\partial s^2 = -2(1+r)\rho A^s/(r-s)^3$. It is easy to check $\partial L^s/\partial s > 0$ and $\partial^2 L^s/\partial s^2 < 0$ for $\rho \in (0, \rho^F)$, and for $\rho < 0$, $\partial L^s/\partial s < 0$ and $\partial^2 L^s/\partial s^2 > 0$. ■

Proof of Proposition 6. For $\rho \geq \rho^F$, a downward-sloping L^d and a perfectly elastic L^s ensure the existence and uniqueness of equilibrium s^* with $n^* = N$, hence (i). For $\rho = 0$, assets are equivalent to money with zero inflation, and the proof follows Proposition 4. For $\rho \in (0, \rho^F)$, L^d and L^s intersect once and there exists a unique equilibrium. For $\rho < 0$, $\bar{s}^C \geq r$ according to Lemma 3. If $\bar{s}^C = r$, $\nexists$ non-degenerate equilibrium; if $\bar{s}^C > r$, L^d and L^s may have more than one intersection, hence more than one candidate equilibrium. Given n^* being a function of s, we can rewrite the seller's problem (24) as

$$\max_s \alpha\left(n^*(s)\right)\left[\frac{\alpha\left(n^*(s)\right)u - n^*(s)\,\Omega}{\alpha\left(n^*(s)\right) + n^*(s)\,s} - c\right].$$

Given different values of s^* satisfying the first-order condition, some are local minimizers and some are local maximizers. Following Gu and Wright (2016), we can show that under generic ρ, the global maximizer, hence the equilibrium, is unique. Next is to show the existence of $\underline{\rho}$. If $\bar{s}^C = r$, $\underline{\rho} = 0$. Consider $\bar{s}^C > r$. $s \leq r$ implies $\rho \geq 0$, and thus (iii). For $s \in (r, \bar{s}^C)$, $\rho < 0$, $\partial L^s/\partial \rho = (1+r)A^s/(r-s) < 0$, and L^d is constant. Hence, $\exists!\ \rho^*(n)$ such that $L^s(\rho^*) = L^d$, and define $\underline{\rho} = \min_{s\in[r,\bar{s}^C]} \rho^*(s) < 0$. For $\rho < \underline{\rho}$, $L^s(\rho) > L^d$, and there exists no equilibrium, hence (iv).

For the rest of the proposition on participation and asset prices, first consider $\rho \geq \rho^F$. According to Lemma 4, the cost of holding assets $s = 0$, implying $\varphi = \varphi^F$ and $n^* = N$. Let $\rho^N = (r - s_N)p^c/(1+r)A$. If $\rho \in (\rho^N, \rho^F)$, then $s^N > s > 0$. The buyer's participation constraint is slack, and $(\varphi + \rho)A^s/N = p^c$. Hence, $n^* = N$ and $\varphi = \varphi^N = (1+s)p^c/(1+r)A > \varphi^F$. If $\rho \in [\underline{\rho}, \rho^N]$, the buyer's participation constraint is binding, and $s > 0$ and $(\varphi + \rho)A^s/n^* = p^c$. Therefore, $\varphi = \varphi^{n^*} = n^*(1+s)p^c/N(1+r)A > \varphi^F$. ■

Proof of Proposition 7. The Kuhn-Tucker conditions are

$$
\begin{aligned}
0 &= -\eta\left(\tau u-p\right)^{\eta-1}\left(p-\tau c\right)^{1-\eta}+\left(1-\eta\right)\left(\tau u-p\right)^{\eta}\left(p-\tau c\right)^{-\eta}-\lambda_1 && (27)\\
0 &= \eta u\left(\tau u-p\right)^{\eta-1}\left(p-\tau c\right)^{1-\eta}-c\left(1-\eta\right)\left(\tau u-p\right)^{\eta}\left(p-\tau c\right)^{-\eta}-\lambda_2 && (28)\\
0 &= \lambda_1\left(D-p\right),\, 0=\lambda_2\left(1-\tau\right),
\end{aligned}
$$

where λ_1 and λ_2 are the multipliers for p and τ. If $\lambda_1 = 0$, $p^B = (1-\eta)\,\tau u + \eta\tau c \equiv \tau\bar{p}^B$. Substituting this into (28) implies $\lambda_2 > 0$ and $\tau^B = 1$. We also need $D > \bar{p}^B$ to satisfy $p^B < D$. If $\lambda_2 = 0$, $\tau^B = (\eta u + (1-\eta)c)p/uc$. Substituting into (27) implies $\lambda_1 > 0$ and $p^B = D$. For $\tau^B < 1$ to hold, $D < uc/(\eta u + (1-\eta)c) = \underline{p}^B$. If $\lambda_1 > 0$ and $\lambda_2 > 0$, then $p^B = D$ and $\tau^B = 1$. We need $\underline{p}^B < D < \bar{p}^B$ to guarantee $\lambda_1 > 0$ and $\lambda_2 > 0$. Finally, $D \geq c$ is necessary to yield non-negative surplus from trade for sellers. ■

Proof of Proposition 8. The proof is similar to Proposition 7. ■

Proof of Lemma 5. We have the following Kuhn-Tucker conditions

$$
\begin{aligned}
0 &= -\eta\left(\tau u-p\right)^{\eta-1}\left(p-\tau c\right)^{1-\eta}+\left(1-\eta\right)\left(\tau u-p\right)^{\eta}\left(p-\tau c\right)^{-\eta}-\lambda_1 && (29)\\
0 &= \eta u\left(\tau u-p\right)^{\eta-1}\left(p-\tau c\right)^{1-\eta}-c\left(1-\eta\right)\left(\tau u-p\right)^{\eta}\left(p-\tau c\right)^{-\eta}-\lambda_2 && (30)\\
0 &= \lambda_1\left(\phi_t m-p\right),\, 0=\lambda_2\left(1-\tau\right),
\end{aligned}
$$

with λ_1 and λ_2 being the multipliers on the monetary and lotteries constraint. If $\lambda_1 = 0$, $p^B = \tau\bar{p}^B$. Substituting into (30) implies $\lambda_2 > 0$, and hence $\tau^B = 1$, which requires $\phi m > \bar{p}^B$. If $\tau^B < 1$, then $\lambda_2 = 0$ and $p^B = \tau\underline{p}^B$. Substituting into (29) implies $\lambda_1 > 0$ and $p^B = \phi m$, hence $\phi m < \underline{p}^B$. If $\lambda_1 > 0$ and $\lambda_2 > 0$, $p^B = \phi m$ and $\tau^B = 1$. $\lambda_1 > 0$ implies $\phi m < \bar{p}^B$, and $\lambda_2 > 0$ implies $\phi m > \underline{p}^B$. Finally, there is no trade if $\phi m < c$. ■

Proof of Proposition 9. First, the buyer does not want to bring $\phi_{t+1}\hat{m} > \bar{p}^B$ if $i > 0$, or $\phi_{t+1}\hat{m} < c$ for no trade. Then, for $\phi_{t+1}\hat{m} \in (\underline{p}^B, \bar{p}^B)$, $v'(\hat{m}) = -\phi_{t+1}\alpha(n)/n - i\phi_{t+1} < 0$, and the optimal money holding is $\phi_{t+1}\hat{m} = \underline{p}^B$. For $\phi_{t+1}\hat{m} \in (c, \underline{p}^B)$, $v'(\hat{m}) =$

$\phi_{t+1}[\alpha(n)\eta(u-c)/nc-i]$. Since $\underline{p}^B[\alpha(n)\eta(u-c)/nc-i] = \alpha(n)\left(u-\underline{p}^B\right)/n - i\underline{p}^B$, $v'(\hat{m})$ shares the same sign as $\alpha(n)\left(u-\underline{p}^B\right)/n - i\underline{p}^B$. Suppose $v'(\hat{m}) < 0$, then buyers choose $\phi_{t+1}\hat{m} = c$ and $\tau^B = 0$. Suppose $v'(\hat{m}) > 0$, then buyers with measure n^* choose $\phi_{t+1}\hat{m} = \underline{p}^B$. The cutoff i satisfying $v'(\hat{m}) = 0$ is $\bar{\imath}^B = \lim_{n\to 0}\alpha(n)(u-\underline{p}^B)/n\underline{p}^B = \eta(u-c)/c$. Therefore, if $i < \bar{\imath}^B$, $\exists!$ SME with $\phi_{t+1}\hat{m} = \underline{p}^B$ and $\tau^B = 1$; otherwise, there is no monetary equilibrium. Define $i^N = \alpha(N)(u-\underline{p}^B)/N\underline{p}^B = \alpha(N)\eta(u-c)/Nc < \bar{\imath}^B$. If $i \le i^N$, $n^* = N$; otherwise, $n^* < N$. ■

Proof of Proposition 10. We need to check that sellers always post $\tau^C = 1$ and the rest of the proof follows Proposition 4. Let λ be the multiplier for τ, and the FOCs are

$$\begin{aligned} 0 &= \varepsilon(n)(p-\tau c) - \frac{\alpha(n)[1-\varepsilon(n)](\tau u - p)}{\alpha(n)+ni}, && (31)\\ 0 &= \tau\left[\frac{\alpha^2(n)u}{\alpha(n)+ni} - \alpha(n)c - \lambda\right], && (32)\\ 0 &= \lambda(1-\tau). \end{aligned}$$

Given the buyer's optimal participation $n = n^*$ and (31), we have

$$p^c = \frac{\alpha(n^*)\{[1-\varepsilon(n^*)]\tau u + \varepsilon(n^*)\tau c\} + \varepsilon(n^*)n^* i\tau c}{\alpha(n^*) + \varepsilon(n^*)n^* i}.$$

Solve for λ from (32), and we need $\lambda = \alpha(n^*)(u-c) - cn^*i > 0$ to assure $\tau^C = 1$. Since $p^c/\tau > c\ \forall\tau$, $\alpha(n^*)(u-c) - cn^*i > \alpha(n^*)(u - p^c/\tau) - n^*ip^c/\tau \ge 0$. The last inequality is the buyer's DM participation constraint, which holds if $i \le \bar{\imath}^C$ and $n^* > 0$. ■

Proof of Lemma 6. The proof is similar to Lemma 5. ■

Proof of Proposition 11. The proof of equilibrium existence and uniqueness directly follows Proposition 9. Notice the cutoff spread satisfying $v'(\hat{a}) = 0$ is given by $\alpha(n)(u - \underline{p}^B)/n - s\underline{p}^B = 0$, equivalent to the participation constraint $n[\alpha(n)\beta(u-\underline{p}^B)/n - (1-\beta)\underline{p}^B]/A^s = g(n) \ge -\rho$. Since $g''(n) < 0$, let $\underline{\rho} = -\max_n g(n)$, $\rho^F = (1-\beta)\underline{p}^B/A$, and $\rho^N = [(1-\beta)\underline{p}^B - \beta\alpha(N)(u-\underline{p}^B)/N]/A$. For $\rho \ge \underline{\rho}$, all equilibria feature $p^B = \underline{p}^B$ and $\tau^B = 1$. If $\rho \ge \rho^F$, then $\varphi = \varphi^F$; otherwise $\varphi > \varphi^F$. If $\rho \ge \rho^N$, then $n^* = N$; otherwise $n^* < N$. The rest of the proof on equilibrium stability follows Proposition 5. ■

Proof of Proposition 12. The proof is similar to Proposition 10, with the cost of holding assets being s instead of i. ■

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