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Towards an Economics of Irrigation Networks

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Abstract

Both the economics and the engineering of irrigation design are typically based on the assumption of a single source. The more general economic problem is to determine which sources should be developed and how water should be allocated and delivered to various receptor-farmers. This is a problem in network economics. We begin our exploration with the problem of allocating irrigation water from existing sources when the conveyance structures are already in place. Transporting water from a particular source to a farmer entails a conveyance loss such that only a fraction of water sent from the source is received by the farmer. Optimal allocation requires that irrigation demands are matched with the least-cost source, including conveyance losses. Economic networks are then defined as optimally-matched subnetworks. Allocation within each economic network is then determined by equalizing the marginal products of water across farmers, reckoned at the source. Different cases are considered depending whether the sources have similar or different cost functions. We provide a modest beginning to the problem of endogenous sources by examining the problem of locating a single source within the network. We also provide a possible reconciliation of equity and efficiency objectives.

Keywords: Water Networks, Spatial Efficiency, Conveyance Losses.
JEL Classification Q25 · D85

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1 Introduction

According to the 2012 UN World Water Development Report (WWDR), there will be a sharp growth in demand for water resources in the next two decades: food demand is predicted to increase by 50% and demand for energy, which includes hydropower, is predicted to increase by 60% (WWAP [19]). Inasmuch as water is fungible, pressure from these various sectors is said to require an estimated additional 6% increase in water withdrawals (FAO [10]). Faced with the prospect of an impending global water crisis, governments, water authorities, and academics are trying to find innovative ways of managing water resources.

The challenge, therefore, is to efficiently allocate water across time and space. Modeling water distribution along these dimensions is complicated by considerations such as conveyance losses, water recharge to the aquifer, and the like. Several studies that explored efficiency in a dynamic setting include Roumasset and Wada [22] where they examine the optimal path of groundwater extraction in the face of extraction costs and groundwater recharge. If extraction costs were sufficiently convex, then the extraction path of groundwater will approach a steady state above the maximum sustainable yield level of groundwater stock. This line of research in efficiently allocating water across time has provided insights on the optimal ordering of extracting water resources such as groundwater, recycled water and desalination (see Roumasset and Wada [23]).

On the other hand, several papers dealt with water allocation across space. For instance, Chakravorty and Roumasset [7] develop a spatial model of irrigation in the presence of conveyance losses. In their linear network model, they find that efficient water distribution implies that farms farther away from the source receive less water and that farmers at the tail end of the system may be required to pay a larger total charge for receiving less water[1]. In another paper,

[1]Charging farmers the wholesale marginal cost reckoned at the headworks means that tail farmers

Chakravorty, Hochman and Zilberman [6] model the choice of reducing these conveyance losses through investment, for instance, in improved canal lining or other conservation technology.

In this context, network economics may provide a useful approach along several dimensions. First, since most of the recent studies that deal with water allocation issues assume that the water network is already known (that is, the connection of sources to farms is given), network economics may extend conventional principles of irrigation design. For instance, the previously mentioned studies on spatial efficiency only deal with a single water source with farms arrayed in a linear network. In a model with several alternative water sources (e.g. multiple dams, lakes, and/or availability of deep and shallow tubewells), efficiency should require some knowledge of which of these sources should be developed and how they should or should not be linked together. Certain principles from network economics may provide a guide to achieve desirable characteristics of water distribution, such as efficiency, over a general network setting. Second, since engineering design principles of water networks do not fully take into account optimal allocation, allocation principles obtained from network economics may help to increase the net present value of these irrigation projects.

There is some literature on water networks, specifically on cost sharing. Aadland and Kolpin [1] discussed the appeal of several cost sharing variants of serial and average cost sharing in twenty-five irrigation ditches located in Montana. They provided an axiomatic characterization of these mechanisms that underpinned the attractiveness of these cost-sharing mechanisms. Márkus, Pintér and Radványi [16] discussed a solution concept in cost sharing called

pay more for each unit of water received. Since optimality typically requires more water sent to tail farmers even though less water is received, this implies that tail farmers pay more for less. That is, full marginal cost pricing may require tail farmers to pay a greater total fee even though they receive less water than head farmers.

the Shapley value in a class of "irrigation games". There is also a strand in this literature that discusses ways to divide a cost of maintaining a waterway (say, a river) with several agents benefitting from that waterway (see Ambec and Sprumont [3], Ambec and Ehlers [2], Ni and Wang [18] for the context of a linear network and Dong, Ni and Wang [9] for the context of a general network). There is also a related problem in graph theory where we want connect a set of agents to a single source at the minimal cost. Several algorithms have been proposed to find the minimal cost spanning tree of a connected network. The minimal cost spanning tree is the network that connects all the nodes together at the least cost (see Bird [5], Bergantiños and Vidal-Puga [4] and Claus and Kleitman [8]). A related problem in this regard is to efficiently select a location for a water source. There are also many rules or algorithms proposed in the literature that cover this class of problems (see, for instance, Vygen [24]).

In this paper we aim to provide principles for efficient allocation of water over a network and to illustrate some algorithms to operationalize those principles. It takes into account transportation or conveyance losses in bringing water from the source to the farm. In several cases developed throughout the paper, we examine how marginal costs of holding water at the source[2] and conveyance efficiency affect the efficient distribution of water in the network.

We develop the spatial model of a water network in the next section. The third section examines the case of a linear network with one source. The fourth section extends to the case of a general network with identical costs for the different sources. The fifth section looks at the case when the sources have different cost functions. The sixth section looks at the related problem in locating a source among several potential locations that will achieve the highest social net benefit. The seventh section reconciles equity and efficiency objectives

[2]This may include the implicit rental cost of capital and operation costs.

in water allocation. The final section concludes the paper and presents some open questions for further research.

2 The model

A water network consists of sources and users with links or connections between them[3]. Examples of sources of water are dams, deep tubewells, and other such structures. Users can be farms or households. Each of the k sources holds a stock of water, Z_k, that can be released to the different farms that are connected to them. There is some water lost in conveyance from source to the farm due to seepage, evaporation and percolation.[4] That is, when source k releases Q_i^k units of water, farm i only receives $q_i^k = Q_i^k h(d_{ik})$. The function $h(d_{ik})$ denotes the percent of water from source k designated for farm i which actually reaches that farm. This function is decreasing in distance d_{ik} (here distance is interpreted to be the distance of the shortest path from source i to farm k), that is, more distant farms require greater amounts of water to be sent in order to receive the same amount of water as nearer farms.

The farms use the water to produce output, which is given by a production function $f(q_i)$, common to all farms (where $q_i = \sum_k q_i^k$). The price of output is given at P per unit. For every extra unit of water the farm receives, we can define the value of how much output is increased. This is given by the *value of marginal product for received water* for farm i from source k:

$$VMPS_i^k(q_i) = Pf'(q_i)h(d_{ik}) \tag{1}$$

[3]We can mathematically represent a water network as a graph consisting of a set of nodes and links.

[4]We consider a typical gravity-irrigation system with a conveyance infrastructure in place. That is, there is no cost of delivering water apart from the water lost en route to evaporation, seepage and percolation.

It is easy to show that the value of equation 1 decreases with distance d_{ik} from the source. The reason is that farms farther away from the source require larger release of water from the source to receive a unit of water. This makes the *VMPS* smaller for farms that are farther away.

The water authority wishes to maximize net social benefits, i.e.,

$$\max_{Q,Z} \left[\sum_{k=1}^{K} \sum_{i=1}^{N} \int_{0}^{Q_i^k} VMPS_i^k(\theta) d\theta \right] - \sum_{k} C_k(Z_k) \tag{2}$$

where $Z_k = \sum_i Q_i^k$ is the total amount of water releases from source k and C_k is the cost function of source k.

In the fourth and fifth sections (exogenous and endogenous matching of sources and demands, respectively) we deal with cases where the cost functions are different between sources. In the section with exogenous matching of sources and demands, we examine the case where marginal cost is constant, that is, the cost of storing an extra unit of water is not increasing. In the section with endogenous matching, we examine the case where marginal cost of storing an extra unit of water is increasing.

In what follows, we provide several interesting network structures to characterize water distribution efficiency.

3 A linear network with one source

This section extends Chakravorty and Roumasset [7] who consider a single source called headworks (e.g. a dam) and farms that are linked along a line with fixed distances between each farm. Consider the network illustrated as in figure 1 and assume at the outset that the demand at each node is known. In this case, the maximization problem in equation 2 simplifies to:

$$\max_{Q,Z} \left[\sum_{i=1}^{N} \int_{0}^{Q_i^1} VMPS_i^1(\theta) d\theta \right] - C_1(Z) \tag{3}$$

subject to

$Z = \sum_i Q_i^1$ where the i's denote nodes instead of farms.

Solving this problem requires constructing the Lagrangian and obtaining the corresponding first order conditions with respect to Q_i^1 and Z. There are two important results that will characterize efficient water distribution in this network. First, the first order conditions imply that

$$VMPS_i^1(q_i^*) = VMPS_j^1(q_j^*) \tag{4}$$

for all farms i,j that receive positive amounts of water.

This equation tells us that for the allocation to be optimal, the value of marginal product of source water should be equal across nodes. If this were not the case, then it would be possible to increase aggregate payoff by reallocating source water to nodes with higher marginal productivity. Note from equation 1 that the only way in which VMPS can be equalized for farms in the network is for the value $Pf'(q_i)$ to increase for nodes that are farther away from the source since the function h is decreasing with distance. This implies that nodes that are farther away should receive less water.

Second, efficient allocation of water should equate the value of marginal product of source water to the marginal cost of producing the water. That is,

$$VMPS_i^1(q_i^*) = C_1'(Z) \tag{5}$$

Farms receiving zero water must satisfy $VMPS_i^1(q_i^*) \leq C_1'(Z)$. That is, the benefit of sending marginal unit of water to farm i is less than the cost of producing the same unit. The first order conditions also imply that $VMPS_i^1(q_i^*) > VMPS_j^1(q_j^*)$ if farm i receives a positive amount of water and farm j does not.

If there are $n \leq N$ farms receiving positive amounts of water, this gives us n equations and n unknowns. This result says that the water authority should equate the extra benefit of releasing a unit of water against the cost of "producing" it. If the former were higher than the latter, then it would make sense for the water authority to increase the amount of water to be stored and released, since the cost of doing so is less than the benefits.

Equations 4 and 5 characterize the *equimarginal principle* of efficiency. Thus, for this case an allocation that solves the water authorityÕs problem in 3 is efficient if and only if it satisfies equations 4 and 5, where the $VMPS$ are equal for farms being served and the $VMPS$ should be equal to the marginal cost.

Consider a numerical illustration where marginal cost is constant and the distances between nodes/farms is given in figure 1.

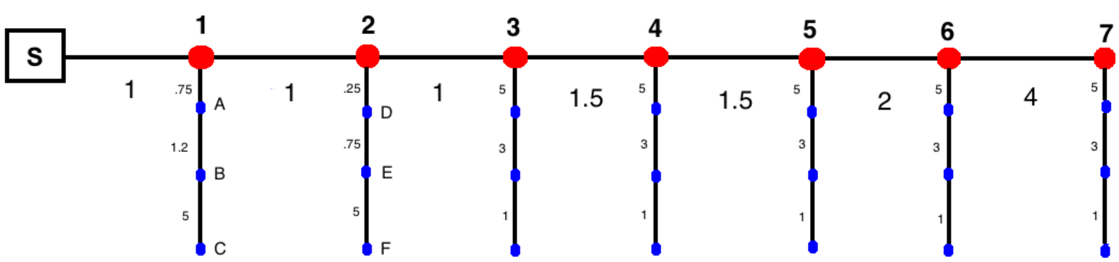

Figure 1: Linear network with one source and seven farms

Following Chakravorty and Roumasset [7], suppose that $h(d_{i1}) = e^{-.02d_{i1}}$ where $e \approx 2.718$ and d_{i1} is the distance of the farm from the source. The numerical value of h is given in the second column in Table 1. The VMPS function is given by $VMPS_i^1 = (10 - q_i)h(d_{i1})$ and the marginal cost function

Table 1: Summary of results from the linear network in figure 1

Farm	h	Optimal q	Optimal Q	VMPS	Benefit	Consumer Surplus
1	0.98	0.82	0.83	9.00	7.85	0.33
2	0.96	0.63	0.66	9.00	6.13	0.20
3	0.94	0.44	0.47	9.00	4.34	0.10
4	0.91	0.15	0.17	9.00	1.51	0.01
5	0.89	0.00	0.00	8.87	0.00	-
6	0.85	0.00	0.00	8.52	0.00	-
7	0.79	0.00	0.00	7.87	0.00	-
Total		**2.05**	**2.13**		**19.82**	**0.64**

is given by $C_1'(Z) = 9$. The table below gives the values of the functions for each farm and summarizes the results.

In this example, only farms 1 to 4 receive an allocation of water from the source while farms 5, 6 and 7 receive no water. Here the farms that received water have their $VMPS$ equal to each other (which is also equal to the constant marginal cost of 9) and the $VMPS$ of the farms that receive zero water is less than the VMPS of the farms that receive a positive amount of water. Moreover, farms that are farther away from the source receive less water.

An important thing to note is that if we assume demands to be nonlinear and cost functions to have some flat parts in relevant ranges, then it may be possible for the sent water (Q) to be increasing with distance but the received water (q) will remain to be decreasing with farms farther from the source (Roumasset and Chakravorty [7]).

Now suppose that MC is non-constant, say increasing in the relevant range. In this case, one cannot simply equate the $VMPS$ function for each farm with the constant MC and solve independently for each Q_i. Rather we have n equations and n unknowns that must be solved simultaneously. One solution procedure is to equate the aggregate demand for water as a function of price with the increasing MC function. This yields the optimal MC and aggregate quantity. Then setting each $VMPS$ to the optimal MC yields the optimal allocation of sent water among farms.

So far we have assumed that we know the demand curves at each node. In order to determine those demands for the case where there are farmers along the secondary canals as illustrated in figure 1, we know simply apply the procedure for determining the aggregate demand at the source, as just described, except this time we treat each node as a source for the farms along the secondary canals.

As an illustration, suppose that the conveyance efficiency along main canal is the same as above and the conveyance efficiency for the secondary canals is given by the formula $h(d_{i0}) = e^{-.02d_{j0}}e^{-.043d_{ij}}$.[5] Thus, the percentage of water that reaches farm i from the source is just the water that remains after it has reached node j (when travelling through the main canal) net of water lost as it travels from node j to farm i in the secondary canal. Thus for node 1 this is equal to $h_1 = [0.95, 0.90, 0.73]$ for farms A, B and C, respectively, and $h_2 = [0.95, 0.92, 0.74]$ for farms D,E and F (ignore nodes 3,4,5,6,7 since in this example farms in these nodes will not receive water). Assume that the $VMPS$ is the same as above and the marginal cost function is $C'(Z) = 9$. Applying the same procedure as described above, the efficient allocation implies that the first two farms from node 1 receive positive amount of water ($q_A = 0.52$ and $q_B = 0.02$, respectively) and also the first two farms in node 2 ($q_D = 0.34$ and $q_E = 0.03$). Notice that even though Farm B is closer to source 1 in terms of distance than farm E, the severity of losses in the secondary canals implies that more water is allocated to farm E since the distance that the water travels through the secondary canals is shorter.

[5]Note that the losses differ for the main and secondary canals, and we assume that there is a higher proportion of water lost per unit distance in the secondary canals, e.g. because of less canal lining.

4 Multiple-source networks: exogenous matching of sources and demands

In this section, we extend the case of the linear network into more general networks. Suppose that there are more than one source and farms are connected to different sources via a water network. Figure 2 gives a simple example of a network with two sources and three farms. This network gives all the potential paths for which water can flow from the sources to the farms. It is useful then to distinguish between this "potential water network" and the "economic water network" which contains only the optimal links where water flows. An economic water network may be the whole potential network or a distinct subnetwork. As discussed in the section with endogenous matching of sources and demands, all the sources in an integrated economic network have the same optimal marginal costs and all the nodes have the same $VMPS$.

If an economic network contains one or more subnetworks, then the marginal costs within each subnetwork equal the $VMPS$ but the marginal costs across subnetworks are unequal. As an illustration, the arrows in Figure 2 represent optimal water flows. We will show that source 1 serving node 1 and source 2 serving nodes 3 and 2 are distinct economic subnetworks.

Consider the case where water sources have different cost functions. For instance, farms may have a choice between sourcing water from a dam or from deep tubewells located in different parts of the aquifer and those sources will have different cost functions.[6] We assume for now that the marginal cost of producing water is constant for each source, that is, $C_k'(Z_k) = \alpha_k$. This means that the cost of producing water will be constant for each extra unit.

[6]There may be important dynamic considerations that we ignore, such as the depletion of the head level (the distance from some reference point to the top of the water table),that may increase extraction costs.

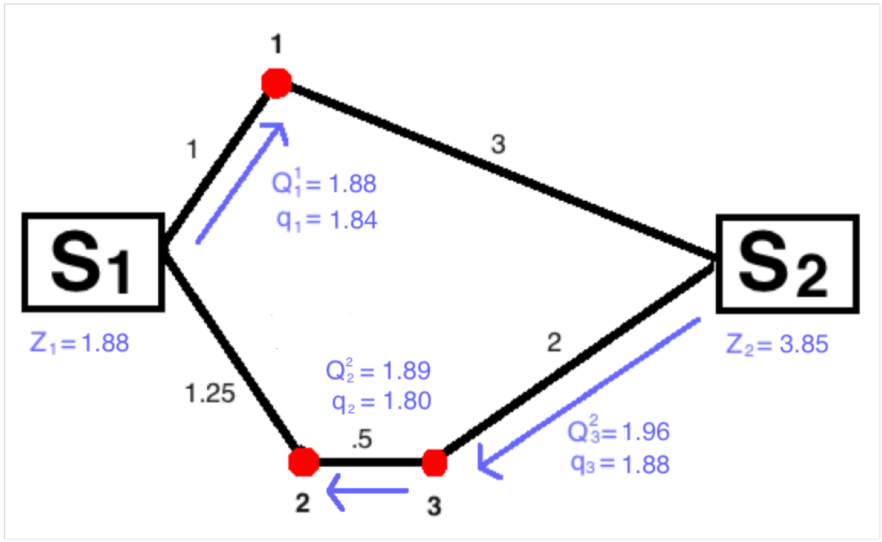

Figure 2: A network with two sources and three farms and two economically integrated subnetworks

The water authority in this setting must determine the optimal production at each source and the allocation to the various farms (including zero from some sources to some farms and zero to some farms). We put forth two principles for determining the optimal allocation. The *least cost principle* states farms must obtain water from the (marginally) cheaper source. The cost of delivered water is the marginal cost at the source plus the unit transport cost. For the case of gravity irrigation where conveyance structures are already in place, this can be measured as the cost of producing one unit of *received* water for each farm, that is, $\frac{\alpha_k}{h(d_{ik})}$. Since marginal costs are constant for each source, the least cost source is exogenously identified by this criterion regardless of the quantities produced by each of the sources. Since there can be different cost functions across sources, the cheapest source may or may not be the one that is closest to the farm. For a set of nodes served by the same source, optimal allocation must also satisfy the *equimarginal principle*, that positive quantities received from the same source must have the same value of marginal products (as reckoned at the source).

With different costs for multiple sources, the water authority's problem now becomes:

$$\max_{Q,Z} \sum_{k} \left[\sum_{i=1}^{N} \int_{0}^{Q_i^k} VMPS_i^k(\theta) d\theta \right] - C_k(Z_k) \tag{6}$$

subject to

$Z_k = \sum_i Q_i^k$

and the non-negativity constraints

$Q_i^k \geq 0$

The first order conditions of equation 6 require that, for each source:

$$VMPS_i^k(q_i^*) = VMPS_j^k(q_j^*) \tag{7}$$

for all farms i,j receiving positive water from source k.[7]

This means that for all agents receiving water from a particular source k, their $VMPS$ should be equal. In other words, in the economic subnetwork generated from the potential network, the $VMPS$ of the farms receiving water from the same source should be equal. Again, if this were not the case, then it is possible to reallocate water from farms with lower $VMPS$ to the ones in the same economics subnetwork with higher $VMPS$ and by doing so increase the total net benefits of the water network.

The first order conditions also yield:

$$VMPS_i^k(q_i^*) = C_k'(Z_k) = \alpha_k \tag{8}$$

[7] In the Appendix we provide a more detailed derivation of the first order Kuhn-Tucker conditions.

for all i farms served by source k.

Equation 8 just says that for a water allocation to be efficient, the $VMPS$ of agents obtaining positive amounts of water from source k should equal the marginal cost of producing Z_k units of water.

Thus, for the allocation of water to be efficient, equations 7 and 8 should be satisfied for farms that receive water from source in a given economic subnetwork.

Let us take the example in Figure 2. Suppose that the first source S_1 costs more to produce an extra unit of water than the second source S_2. For instance, let the cost functions of S_1 and S_2 be $C_1(Z_1) = 8Z_1$ and $C_2(Z_2) = 7.8Z_2$, respectively. Assume again that $VMPS_i^k = (10 - q_i)h(d_{ik})$. Notice that for farms 2 and 3 the closest source in terms of distance is S_1 (1.25 versus 2.5 and 1.75 versus 2, respectively). However, the advantage of this proximity is washed out by the fact that producing water in this source is more expensive. This is shown in column 6 of the table, which computes the marginal cost of one unit of received water. For farms 2 and 3, the net benefit is higher for source 2 than source 1, while for farm 1 this net benefit is higher in source 1. Therefore, the *least cost principle* suggests that when we consider the different marginal cost of producing and transporting water, farms 1 will be connected to S_1 while farms 2 and 3 will be connected to S_2. The numbers in Figure 2, as well as Table 2 below summarizes the optimal values of q, Q and Z.

From the arrows in Figure 2, we see that in the economic water network S_1 sends water to farm 1 while S_2 supplies water to farms 2 and 3. This can also be seen in the positive entries of q in Table 2 2 for farm 1 from source 1 and farms 2 and 3 from source 2. The total net benefit is 5.08, which is higher than the net benefit that would be obtained if we source water exclusively from S_1 (4.78) or from S_2 (4.87). Notice that the $VMPS$ for each of the farms is equal

Table 2: Least-cost sources: low conveyance costs

From Source 1:

Farm	h	Optimal q	Optimal Q	VMPS	Marginal cost of one unit of received water	Consumer Surplus
1	0.980	1.84	1.88	8.00	**8.162**	16.69
2	0.975	-	-	-	**8.203**	-
3	0.966	-	-	-	**8.285**	-
Total		1.84	1.88			16.69

From Source 2:

Farm	h	Optimal q	Optimal Q	VMPS	Marginal cost of one unit of received water	Consumer Surplus
1	0.942	-	-	-	**8.282**	-
2	0.951	1.80	1.89	7.8	**8.200**	16.38
3	0.961	1.88	1.96	7.8	**8.118**	17.05
Total		3.68	3.85			33.43

Total Net Benefit 5.08

Note: The function h is given by the formula $h(d_{ik}) = e^{-.02d_{ik}}$

to the marginal cost of the source that they are connected to. Furthermore, the farms receive water from the source that provides the least cost, as seen in the sixth column, for instance, farm 2 is optimally served by S_2 since the marginal cost of one unit of received water from S_2 is lower (8.200 vs. 8.203).

To illustrate how the least cost source depends on the conveyance losses as well as production costs, consider the case wherein the distances in the links in Figure 2 were to double (so, for instance, the distance from source 2 to farm 2 is now 5 instead of 2.5). Table 3 below summarizes the results with the new distances. Here we can see that farm 2 is now served by S_1 because the marginal cost of one unit of received water is lower (8.41 vs. 8.62).

5 Endogenous matching of sources and demands

Now consider the case of increasing marginal cost for each of the sources. This can be justified, for example, if dam size is already beyond the point where scale economies have been exhausted. That is, building a dam beyond a cer-

Table 3: Least-cost sources: high conveyance costs

From Source 1:

Farm	h	Optimal q	Optimal Q	VMPS	Marginal cost of one unit of received water	Consumer Surplus
1	0.961	1.67	1.74	8.00	**8.326**	15.33
2	0.951	1.59	1.67	8.00	**8.410**	14.63
3	0.932	-	-	-	**8.580**	-
Total		3.26	3.41			29.97

From Source 2:

Farm	h	Optimal q	Optimal Q	VMPS	Marginal cost of one unit of received water	Consumer Surplus
1	0.887	-	-	-	**8.794**	-
2	0.905	-	-	-	**8.620**	-
3	0.923	1.55	1.68	7.8	**8.450**	14.30
Total		1.55	1.68			14.30

Total Net Benefit **3.87**

Note: The function h is given by the formula $h(d_{ik}) = e^{-.02d_{ik}}$

tain height may entail an increase in average costs per expected quantity of water stored. In this case, the least-cost source depends on the quantities produced by each of the sources and cannot be exogenously identified. Optimal matching must now be determined simultaneously with quantities produced and allocation among farms.

The water authority's problem in this section is similar to the last section. However, in that section where marginal cost is constant, farms will typically always receive water from a unique source unless we are in the trivial case where the cost advantage is exactly offset by the distance of the farm to the source.[8] In this section it will be possible for farms to receive water from multiple sources. Intuitively, the water authority will provide water from a source to a farm until it is still cheap to do so (that is, if marginal costs does not rise too much from the additional unit of water), and probably provide water from another source to that same farm if it is profitable to do so.

For the case where a farm receives water from multiple sources, the first order conditions of the water authority's problem imply that this equation must hold:

[8]This happens when $\frac{\alpha_k}{h(d_{ik})} = \frac{\alpha_m}{h(d_{im})}$ for *all* sources that farm i gets water from.

$$\frac{C_k'(Z_k)}{h(d_{ik})} = \frac{C_m'(Z_m)}{h(d_{im})} \tag{9}$$

for all sources k and m that farm i receives positive amounts of water. The condition tells us that farms receiving water from multiple sources must have the marginal costs of each source (weighted by the conveyance efficiency h of that source) equal to each other.

For instance, consider the same network in Figure 2 and suppose that the marginal costs are given by $C_1'(Z_1) = 2 + 5Z_1$ and $C_2'(Z_2) = 5 + 2Z_2$ for sources S_1 and S_2, respectively. Table 4 summarizes the results if we allocate water to the network under three different scenarios: Scenario 1 is the resulting allocation when farm 2 receives water from both S_1 and S_2; Scenario 2 is the resulting allocation when farm 2 receives water exclusively from S_1 and Scenario 3 is where farm 2 receives water exclusively from S_2 .

Note that in this table, the allocation in Scenario 1 yields the highest net benefits for the water authority. Indeed, the water authority will balance sourcing water from one source or the other depending on the marginal costs of doing so. We can check that the condition in equation 8 is satisfied by plugging in the values of Z_1 and Z_2 (1.36 and 1.79, respectively) into the weighted marginal costs.

Figure 3 visually shows the information contained in Table 4, as well as the flow of water from sources to farms. Thus, in the economic water network, S_1 provides water to farms 1 and a small proportion to farm 2 while S_2 provides water to farm 3 and on to farm 2.

Two sources are in the same network if they optimally serve a common demand. Two demands are in the same network if they are optimally served by sources that are in the same network.

While in the constant marginal cost case it is straightforward to determine

Table 4: Summary of results of the network in figure 2 and with increasing cost functions

Scenario 1: Farm 2 gets water from both sources

Farm	h from Source 1	h from Source 2	Optimal q (Source 1)	Optimal Q (Source 1)	Optimal q (Source 2)	Optimal Q (Source 2)	Benefit
1	0.980	9.42	1.017	1.037	-	-	9.65
2	0.975	9.51	0.316	0.324	0.66	0.69	9.24
3	0.966	9.61	-	-	1.06	1.10	10.05
Total			**1.33**	**1.36**	**1.72**	**1.79**	**28.94**
Total Net Benefit	**9.40**						

Scenario 2: Farm 2 gets water exclusively from Source 1

Farm	h from Source 1	h from Source 2	Optimal q (Source 1)	Optimal Q (Source 1)	Optimal q (Source 2)	Optimal Q (Source 2)	Benefit
1	0.980	9.42	0.717	0.732	-	-	6.92
2	0.975	9.51	0.671	0.688	-	-	6.48
3	0.966	9.61	-	-	1.51	1.58	14.00
Total			**1.39**	**1.42**	**1.51**	**1.58**	**27.40**
Total Net Benefit	**9.15**						

Scenario 3: Farm 2 gets water exclusively from Source 2

Farm	h from Source 1	h from Source 2	Optimal q (Source 1)	Optimal Q (Source 1)	Optimal q (Source 2)	Optimal Q (Source 2)	Benefit
1	0.980	9.42	1.283	1.309	-	-	12.01
2	0.975	9.51	-	-	0.84	0.88	8.06
3	0.966	9.61	-	-	0.93	0.97	8.89
Total			**1.28**	**1.31**	**1.77**	**1.86**	**28.96**
Total Net Benefit	**9.34**						

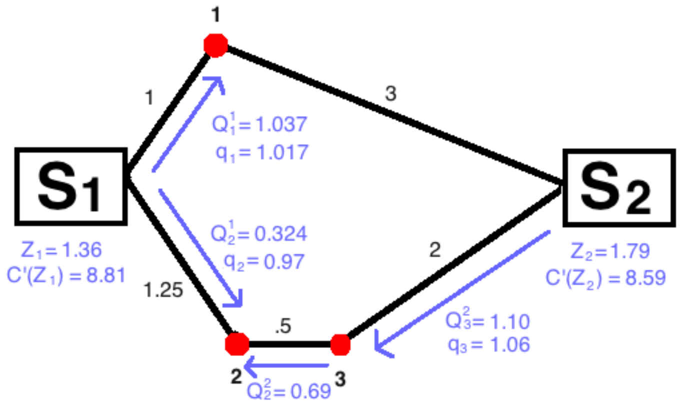

Figure 3: A network with two sources and three farms

the least cost source from which a farm will obtain water, it is more complicated in the case of increasing marginal costs. One straightforward algorithm to solve this is to articulate all possible economic subnetworks of a given potential water network and to compute the allocation that will yield the maximum net benefit for each of this subnetworks. Of course as the number of farms and sources increases, finding the optimal design belongs to the class of "NP-hard" problems.[9] That is, computations using this algorithm become practically infeasible as the number of sources and farms increases.[10] The development of algorithms to determine the least cost source in larger networks is a matter of ongoing research in network economics.

6 Endogenous location of sources

Consider a water authority that faces substantial fixed cost that limits its ability to build multiple water sources. Thus, a related problem of efficient water distribution is where to place a source among potential locations for these

[9]See, e.g. Garey and Johnson [11].

[10]In practice, computer scientists often use approximation theory to obtain good, not perfect, solutions.

sources. This section focuses the discussion on the problem of locating one source out of several locations and that the cost of building the infrastructure is the same across all locations. For example, suppose that only one of the two potential sources illustrated in Figure 2 can be developed, i.e. either S_1 or S_2.[11]

In principle, we want to place our source in the location that maximizes the social net benefit. In doing so there are two things to consider. First, we have to consider the distance of each farm from the source because the farther away farms are from it the less water it receives, which in turn lowers the payoff to the farm. Second, we have to consider the cost of producing water. If marginal costs are sufficiently low such that it is optimal to send positive amounts of water to all farms, then we only have to consider the efficiency of transporting the water to these farms. However, when marginal costs are high, not all farms will be served and the water authority has to consider only the returns to the farms that are relatively nearer to the source. Indeed if the marginal cost of producing water is close to zero, the problem of locating one source in the water network is equivalent to the problem of finding the source with the highest efficiency, that is, the location which has the highest sum of the function h. The intuition is that since all farms will be served because of the (almost) zero marginal cost, the source that yields the highest sum of h will have the highest percentage of their water released reaching the farms. In the example in Figure 2, if marginal cost were zero then the location of the source will be S_1.

However, if marginal cost were high or rising, then some agents in the water network may not be served. The water authority therefore has to give priority to the payoff to the agents who are closer to the head. This will give the highest social net benefit among all possible locations for the source. As with

[11]These source locations may be multiple upstream reservoirs, lakes and/or deep and shallow tubewells. This implicitly includes the choice between surface water and groundwater.

the previous section on increasing marginal costs, a straightforward algorithm to determine the most efficient source location is to list all possible economic subnetworks of a given potential location and to compute the allocation that will yield the maximum net benefit for each of this subnetworks. But as the number of potential sources and number of farms increases, this "brute force" algorithm may be difficult to implement.

7 Cost Sharing Mechanisms and Equity

7.1 Cost sharing mechanisms

Finally it is interesting to look at cases that examine desirable properties beyond efficiency. We have so far characterized allocations within the water network when we charge marginal cost. However, there may be several problems with marginal cost pricing. In this subsection, we discuss two potential equity problems: First, there may be head-tail inequities inasmuch as consumer surpluses for tail farmers are relatively small. Second, marginal cost pricing may be unnecessarily onerous if it generates a revenue surplus. Johansson et al. [13] document many alternative mechanisms to marginal cost pricing that address other important dimensions of allocation, especially equity concerns. Here we consider fixed fee, equal pricing, and lump sum transfers.

In many countries, farmers are charged a fixed fee, which is below marginal costs. This mechanism suffers from the classic "head-tail" problem where farmers that are closer to the source (the "head" farmers) overdraw water and farmers that are far from the source (the "tail") under-draw water, relative to the efficient solution.[12] This type of inefficiency has a close parallel to the aver-

[12] It should be noted that head farmers using more water than tail farmers is not an *a priori* indicator of inefficiency since this phenomenon is true even in the efficient solution.

age cost sharing mechanism in the mechanism design literature and has been quantified in several network problems.[13]

Another candidate mechanism is *equal pricing* (EP), which maximizes the surplus of the agents subject to the condition that agents pay the same price per unit of received water and the water authority recovers some proportion of total cost. This mechanism generates a price for each farmer that divides the total cost of producing the water with the proportion of his received water to the total amount of received water by all of the farmers. That is, each farmer's total water bill is $\frac{q_i}{\sum_i q_i} C(Z)$. The main advantage of EP is that at the equilibrium the agents with similar utilities will demand the same amount of water in their farms and will be charged the same price. In particular, this implies that agents with similar utilities get the same net benefit (a key property in the fairness literature named *equal treatment of equals*). Using the same parameters as in Table 1, Table 5 below shows the allocation of received water, price per unit of received water and consumer surplus under the EP mechanism are equalized.

The main downside of EP is that it is inefficient relative to the case where we charge marginal cost. In table 5, we can see that agents will receive the same amount of water at their farms. Therefore, agents who are farther away from the source are sent a larger amount of water. This is contrary to the efficient allocation, where agents closer to the source are sent larger amounts of water than agents farther away to the source. The allocation given by EP generates a loss in the consumer surplus equal to .18, which equals a loss of 28% of the net social benefit below the efficient solution.

One way of pursuing equity without sacrificing efficiency is the mechanism of marginal cost pricing with lump sum transfers. For example the head farmers in table 1 could be charged lump sum taxes that are distributed to the tail

[13] See Moulin [17] for a comparison of the average cost, marginal cost and serial cost. See Juarez [14] for a comparison of the average cost relative to the random priority mechanism.

Table 5: A mechanism that charges the served farmers the same price per unit of received water

Farm	h	q	Q	Price	Price per unit of received water	Benefit	Consumer Surplus
1	0.98	0.97	0.99	9.03	9.27	9.26	0.23
2	0.96	0.97	1.01	9.03	9.27	9.26	0.23
3	0.94	0.00	0.00	-	-	-	-
4	0.91	0.00	0.00	-	-	-	-
5	0.89	0.00	0.00	-	-	-	-
6	0.85	0.00	0.00	-	-	-	-
7	0.79	0.00	0.00	-	-	-	-
Total		**1.95**	**2.01**			**18.52**	**0.46**

farmers in accordance with a specific equity criterion. Since transfers are lump sum, they will not affect the optimal quantities of water demanded.

Trying to redistribute towards farmers with lower net benefits is problematic, however. Horizontal equity does not demand redistribution since, from an ex-ante point of view, farmers are typically not equal. Head farmers have the advantage of being close to the source, which confers a natural advantage over tail farmers. And even if the government is pursuing vertical equity, there is no necessary reason for building redistribution into each and every project. The government is likely to have more efficient mechanisms for addressing basic needs and redistributing income than burdening individual projects with those requirements (Kaplow and Shavell [15]). This does not mean, however, that equity and efficiency are incompatible in irrigation pricing.

Another drawback of marginal cost pricing is that it may leave a revenue surplus above the appropriate level of cost recovery. First, marginal cost may be rising in the relevant range such that marginal cost pricing leaves a surplus above full costs. Second, under the principle of benefit taxation, direct-beneficiary farmers should only be responsible for the proportion of costs equal to the ratio of direct to total benefits. Indirect beneficiaries may include commercial interests who benefit from increased agricultural production and consumers who benefit from lower prices (Roumasset [20], [21]). One mechanism for limiting charges to direct beneficiaries while maintaining the efficiency property of marginal cost pricing is the lump sum transfer approach described above. Here we consider a particular form of lump sum transfers, a block pricing scheme (BP mechanism) that leaves farmers with the same ratio of charges to benefits. There are several steps involved in BP: First, the water authority must identify the revenue target, that is, the percentage of total cost that should be recovered from direct beneficiaries. Second, the revenue requirement must be allocated among individual farmers. The BP mechanism charges every served farmer a first tier price at which they will be entitled to draw a certain amount of (sent) water, Q_i^e. For quantities exceeding the first block, prices are set equal to the constant marginal cost. The mechanism sets the quantity of the first block such that the ratio of costs to benefits is equal for each farmer and the revenue collected from direct beneficiaries is equal to the target revenue. Figure 4 illustrates. In figure 4, the i^{th} farmer pays the horizontally-shaded area in exchange for total benefits equal to the sum of the horizontally and vertically-shaded areas.

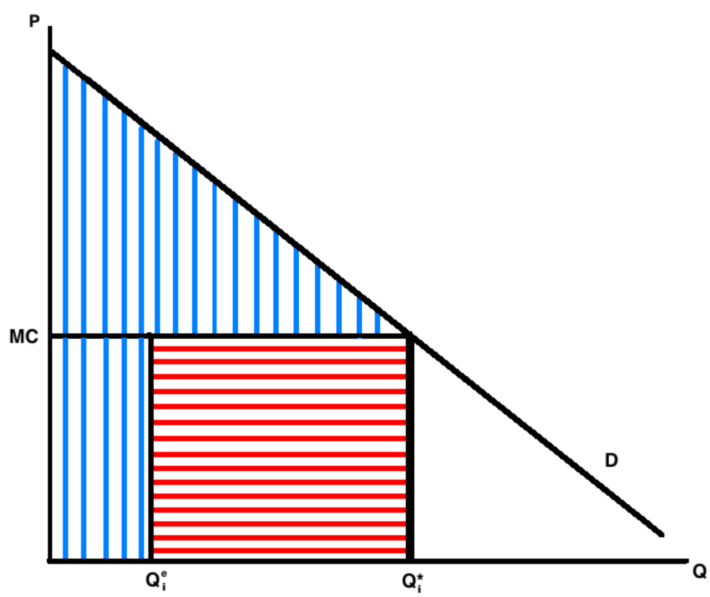

Figure 4: The block pricing mechanism

In order to compute for the entitlements and the charges under the BP mechanism, the following steps can be followed. First, we determine the indirect benefits at the optimal solution, IB^*, through a cost-benefit exercise. Given IB^* and the total benefits at the optimal solution,DB^*, we can compute the revenue target by using the formula $RT^* = \frac{DB^*}{DB^*+IB^*} \times MC \times Z^*$, where MC is the marginal cost and Z^* is the total optimal amount of sent water. The proportion of total benefits that will satisfy the revenue requirement is therefore $\pi^* = \frac{MC \times Z^*}{DB^*+IB^*}$. To compute each farmer's entitlement, we then solve for Q_i^e in: $\frac{MC\left(Q_i^* - Q_i^e\right)}{B_i} = \pi^*$ where Q_i^e is the entitlement for farmer i, Q_i^* is the optimal amount of sent water to farm i, and B_i is the benefit for farm i (area under the demand curve to the left of Q_i^*).

Table 6 provides an illustration of the BP mechanism based on the parameters in Table **??**. Suppose that indirect benefit is $IB^* = 5.52$. Given that $IB^* = 19.82$, $MC = 9$ and $Z^* = 2.13$, the revenue target is $RT^* = \frac{19.82}{19.82+5.52}(9 \times 2.13) = 15$. The proportion of total direct benefits that satisfies the revenue target given DB^*, IB^*, MC and Z^* is then $\pi^* = 0.76$, which is given in column 6 of table 6. Given π^*, MC, Q_i^* in column 4 and the direct benefits B_i in column 5 of table 6, we are able to obtain the entitlements and the corresponding charges for each of the served farmers by the formula $\frac{MC\left(Q_i^* - Q_i^e\right)}{B_i} = 0.76$. The list of entitlements to each farmer is given in column 7 while the charges are given in column 8. Accordingly, the individualized entitlements achieve proportional benefit taxation for all farmers. In practice, the entitlements might be set according to farmer category to economize on administrative costs. Higher indirect benefits, lower MC, and higher water demands would all lead to higher farmer surpluses. Since the farmer subsidies implicit in the entitlements are inframarginal, they are lump sum transfers and do not distort the quantity decisions.

Table 6: Block pricing for a revenue target of 15

Farm	h	Optimal q (q^*)	Optimal Q (Q^*)	Benefit B_i	Ratio of Charges to Benefits (π^*)	Entitlements (Q_i^e)	Charges [$MC \times (Q^* - Q^e)$]
1	0.98	0.82	0.83	7.85	0.76	0.17	5.94
2	0.96	0.63	0.66	6.13	0.76	0.14	4.64
3	0.94	0.44	0.47	4.34	0.76	0.11	3.28
4	0.91	0.15	0.17	1.51	0.76	0.04	1.14
5	0.89	0.00	0.00	0.00	-	-	-
6	0.85	0.00	0.00	0.00	-	-	-
7	0.79	0.00	0.00	0.00	-	-	-
Total		**2.05**	**2.13**	**19.82**			

Note: See text on the computation of columns 6 to 8. Marginal cost is equal to 9.

8 Compensation mechanisms

In the above setting where the source is known, farmers are inherently unequal by virtue of their locations relative to the source. In contrast, consider an endogenous facility-location problem wherein agents are uniformly distributed in a linear network and there are two potential equal-cost facilities at each extreme, only one of which can be developed (for example due to fixed costs). Picking the facility at one extreme will benefit agents who are closer to that facility and harm the agents who are closer to the other extreme of the network.[14] In this case, the BP mechanism would violate equal treatment of equals and some additional compensation may be appropriate.

To see this, suppose that we have three farms and two potential sources. Farm 1 is closest to source 1 and farm 3 is closest to source 2 while farm 2 is in the middle of farm 1 and 3. The distance between each farm is 1 kilometer, and the distance from source 1 to farm 1 and from source 3 to farm 3 is likewise 1 kilometer. Assume that $VMPS_i = (10 - q_i)h(d_{ik})$ for all farms and each source has the same marginal cost, that is, $C_k(Z_k) = 9.5$ for both potential sources 1 and 2. Assume the h function is the same as that in the section dealing with a linear network. For the sake of transparency, assume also that there are no indirect benefits such that inframarginal block pricing is not called for. Table 7

[14]This harm might be coming due to the loss in value of the land and will also result in an unequal distribution of the benefits, since the agents closer to the facility will get a larger benefit.

Table 7: Optimum values in the symmetric example if source 1 or source 2 is chosen

From Source 1:

Farm	h	Optimal q	Optimal Q	VMPS	Benefit	Consumer Surplus
1	0.98	0.31	0.31	9.50	3.03	0.05
2	0.96	0.11	0.12	9.50	1.12	0.01
3	0.94	-	-	9.42	-	-
Total		**0.42**	**0.43**		**4.15**	**0.05**

From Source 2:

Farm	h	Optimal q	Optimal Q	VMPS	Benefit	Consumer Surplus
1	0.94	-	-	9.42	-	-
2	0.96	0.11	0.12	9.50	1.12	0.01
3	0.98	0.31	0.31	9.50	3.03	0.05
Total		**0.42**	**0.43**		**4.15**	**0.05**

shows the optimal amounts if source 1 and source 2 is chosen.

From the point of view of the water authority in this example, it does not matter whether to build source 1 or source 2 since it yields the same net social benefit. However, from an equity point of view, the choice whether to build source 1 or source 2 has implications on the welfare of farmer 1 and farmer 3. Indeed, if source 1 is built then farmer 3 gets no allocation of water and hence no consumer surplus. But if source 2 is built then farm 3 now gets a positive allocation and farm 1 receives nothing.

A simple compensation mechanism is to implement lump sum transfers that leave every agent with the same benefits regardless of which project is chosen. For instance, if there were an equal chance of either source to be built in Table 7, then the lump sum tax that would take away .025 units of consumer surplus from farmer 1 if source 1 is built and transferred to farmer 3. This simple scheme thus leaves all farmers indifferent between the development of source 1 or source 2. (Farmer 2 is already indifferent without transfer.) In this example, this compensation mechanism satisfies equal-treatment-of-equals. This same principle can be generalized to non-symmetric distances, general network structures and different probabilities of projects being implemented by designing transfers that leave farmers indifferent between sources developed.

9 Concluding remarks and some open questions

This paper has demonstrated how network economics can be a useful approach generalizing principles of efficient and equitable irrigation development. For the case of given headworks and canals, we first determine optimal matching between sources and farmers by the principle of least cost in terms of water received. If farms optimally get water from the same source according to the least cost principle, then they are in the same economic subnetwork. The equimarginal principle then says that water should be allocated among farmers in the same economic subnetwork such that the values of marginal products reckoned at the source ($VMPS$) are equal. In case only one source can be developed as in the section dealing with the endogenous location of sources, one applies the equimarginal principle to each potential source and chooses the source with the highest resultant welfare. Where volumetric pricing is feasible and farm productivities can be estimated, there is no need to compromise efficiency for equity inasmuch as lump sum transfer can achieve the equity objective in question without violating marginal cost pricing. Block pricing may be a convenient way to implement said transfers. We believe that future research is warranted to generalize these principles to the case of many sources and farmers with fully endogenous development of sources and canals.

The principles derived in this paper can be extended in several ways. First, a water basin may contain possible sources that could potentially be developed, for example, a lake, different points in a river, deep tubewells, shallow tubewells and rainfed reservoirs. Which of these should be developed cannot be determined by cost-benefit analysis of each individual source. Rather the gain from adding a particular source depends on what other sources and transmission networks might be developed. Network economics can also be extended across inter-basin water transfers.[15] This involves more than just a comparison of two networks with and without a connection. As with sources within a basin, connecting sources has implications for the transmission network and water allocation. A second extension involves dynamics, especially involving conjunctive use. As detailed in another chapter (Jandoc et al. Chapter XX in this Handbook), which source is used to deliver water to a particular farm changes over time. For example, as groundwater scarcity increases, then the economic network may also change. This has implications, in turn for both allocation and optimal connectivity.

Applying and extending these principles may be facilitated by network economists working together with engineers and computer scientists, starting with a small number of endogenous sources and possible connections to farmers. Together, we believe they can advance a new generation of irrigation engineering.

[15]For a description in inter-basin transfers, see e.g. Ghassemi and White [12]

10 Appendix

In this Appendix, we derive the first order conditions of the problem given in equation 6. Recall that the problem is:

$$\max_{Q,Z} \sum_k \left[\sum_{i=1}^{N} \int_0^{Q_i^k} VMPS_i^k(\theta) d\theta \right] - C_k(Z_k)$$

subject to

$Z_k = \sum_i Q_i^k$

and the non-negativity constraints

$Q_i^k \geq 0$

The Lagrangian is given by:

$L = \sum_k \left[\sum_{i=1}^{N} \int_0^{Q_i^k} VMPS_i^k(\theta) d\theta \right] - C_k(Z_k) - \lambda_k \left(Z_k - \sum_i Q_i^k \right)$

Taking the partial derivatives of the Lagrangian over the variables of interest yields:

$$\frac{\partial L}{\partial Q_i^k} = VMPS_i^k - \lambda_k \leq 0, \quad Q_i^k \geq 0, \quad Q_i^k \left(VMPS_i^k - \lambda_k \right) = 0 \tag{10}$$

$$\frac{\partial L}{\partial Z_k} = -C_k'(Z_k) + \lambda_k = 0 \tag{11}$$

If farm i is sent a positive amount of water from source k, that is, if $Q_i^k > 0$ then equation 10 implies that:

$$VMPS_i^k = \lambda_k \tag{12}$$

Together with equation 11, we have

$$VMPS_i^k = C_k'(Z_k) \tag{13}$$

which means that the $VMPS$ of those who received positive amounts of water from source k should equal the marginal cost of source k.

If a farm receives water from multiple sources, then it must be true that equation 13 must hold *for each of those sources.* By definition of VMPS, we can rewrite equation 13 as

$$Pf'(q_i)h(d_{ik}) = C_k'(Z_k) \tag{14}$$

Since $Pf'(q_i)$ is the same for farm i, the first order conditions imply that for any two source k and m that farm i obtains positive amounts water from, the equation

$$\frac{C_k'(Z_k)}{h(d_{ik})} = \frac{C_m'(Z_m)}{h(d_{im})} \tag{15}$$

must hold.

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