University of Hawai`i at Mānoa Department of Economics Working Paper Series

Saunders Hall 542, 2424 Maile Way,
Honolulu, HI 96822
Phone: (808) 956 -8496
www.economics.hawaii.edu

Working Paper No. 12-3R*

Implementing Efficient Graphs in Connection Networks

by
Ruben Juarez and Rajnish Kumar

February 2012
(*revised version of Working Paper No. 10-22)

Implementing Efficient Graphs in Connection Networks*

Ruben Juarez† Rajnish Kumar‡

December 1, 2011.

Abstract

We consider the problem of sharing the cost of a network that meets the connection demands of a set of agents. The agents simultaneously choose paths in the network connecting their demand nodes. A mechanism splits the total cost of the network formed among the participants.

We introduce two new properties of implementation. The first property, *Pareto Nash Implementation (PNI)*, requires that the efficient outcome always be implemented in a Nash equilibrium and that the efficient outcome Pareto dominates any other Nash equilibrium. The *average cost mechanism (AC)* and other asymmetric variations are the only rules that meet PNI. These mechanisms are also characterized under Strong Nash Implementation.

The second property, *Weakly Pareto Nash Implementation (WPNI)*, requires that the least inefficient equilibrium Pareto dominates any other equilibrium. The *egalitarian mechanism (EG)*, a variation of AC that meets individual rationality, and other asymmetric mechanisms are the only rules that meet WPNI and Individual Rationality.

PNI and WPNI provide the first economic justification of the *Price of Stability (PoS),* a seemingly natural measure in the computer science literature but one not easily embraced in economics. EG minimizes the PoS across all individually rational mechanisms.

JEL classification: C70, C72, D71, D85.
Keywords: Cost-sharing, Implementation, Average Cost, Egalitarian Mechanism.

*We would like to thank our advisor Hervé Moulin for his invaluable insights at the various stages of this paper. This paper benefited from the comments of seminar participants at École Polytechnique, Universitat Autónoma de Barcelona, SED conference in Maastricht, Network Economics conference at LSU, SCW conference in Moscow and ACM Conference on Electronic Commerce in San Jose. Critical comments by Gustavo Bergantinos, Anna Bogomolnaia, Francis Bloch, Anirban Kar, Tim Roughgarden, Sudipta Sarangi, Arunava Sen and Katya Sherstyuk. Ruben Juarez appreciates the financial support from the AFOSR Young Investigator Program under grant FA9550-11-1-0173.

†University of Hawaii, Department of Economics, 2424 Maile Way, Saunders Hall 542, Honolulu, HI 96822. E-mail: rubenj@hawaii.edu, Ph: (808) 956-7143.

‡LSU, Department of Economics, 2138 Patrick F. Taylor Hall, Baton Rouge LA 70803 E-mail: rajnishkumar@lsu.edu, Ph: (225) 578-3805

1 Introduction

Network cost-sharing problem

We consider the problem of sharing the cost of a congestion-free network that meets the connection demands of a set of agents. The agents simultaneously choose paths in the network to connect their demand nodes, and a mechanism splits the total cost of the network formed among the participants. This type of problem arises in many contexts ranging from water distribution systems, road networks, telecommunications services and multicast transmission to large computer networks such as the Internet.

A challenge to designing mechanisms arises because a mechanism induces a game among the agents who choose their paths strategically. Therefore, the traditional objectives of the social planner may be conflicting; and thus, it may not be obvious to choose one mechanism over the other. We focus on mechanisms that are efficient, i.e., the ones that minimize the cost of the network formed at the equilibrium of the game when the agents choose their paths strategically. In other words, the planner wants to design a mechanism that implements the efficient graph.

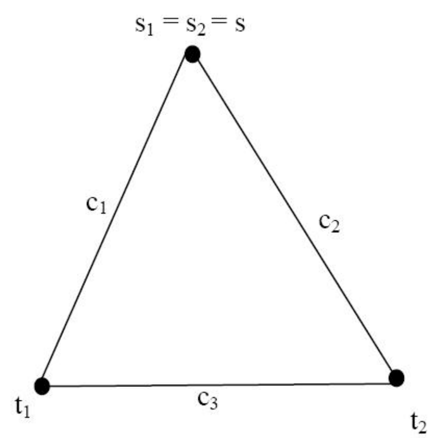

Figure 1: Network with two agents at the common source s and two different sinks t_1 and t_2.

Consider the network in figure 1 with two agents located at the common source s and interested in going to the sinks t_1 and t_2 respectively. The Shapley mechanism (Sh, [6]), which divides the cost of every edge equally across its users, may provide wrong incentives to the players and they may end up choosing an inefficient graph at equilibrium. Indeed, if $c_3 < c_2 < c_1$, then the efficient graph is formed by the links st_2 and t_2t_1, agent 1 chooses the path (st_2t_1) and agent 2 chooses the path (st_2). However, at equilibrium agent 1 does not choose the efficient path whenever $\frac{c_2}{2} + c_3 > c_1$. In general networks, even the best equilibrium of the Shapley mechanism can be as costly as $H(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ times the cost of the optimal graph, where k is the number of users ([2]). Therefore, there is a need to find mechanisms that implement the efficient graph.

Robust efficient implementation

The celebrated literature on full implementation of the efficient outcome has more often than not hit impossibilities (see [29] for a comprehensive survey). In the growing literature in computer science (and more recently in economics[1]), two measures of efficiency loss have been very fruitfully studied. On one hand, there is the traditional price of anarchy (PoA [25]), which computes the ratio of the worst equilibrium over the efficient outcome. On the other hand, there is the price of stability (PoS [2]), which computes the ratio of the best equilibrium over the efficient outcome. Both of these measures have been very effective in selecting second-best mechanisms. However, these approaches lack economic justification and thus seem to be quite arbitrary in the absence of a compelling equilibrium selection rule. The following types of natural questions automatically arise. Why should we study the worst case performance of a mechanism? Why not the best case scenario?

In this paper, we fill this gap by providing new equilibrium selection rules. We introduce two new properties of implementation. The first property, **Pareto Nash Implementation (PNI)**, requires that the efficient outcome always be implemented in a Nash equilibrium and that the efficient outcome Pareto dominates any other Nash equilibrium. Contrary to the traditional literature on full implementation, PNI may implement multiple inefficient equilibria; however, the efficient equilibrium is always implemented and Pareto dominates any other equilibrium.

The second property, **Weakly Pareto Nash Implementation (WPNI)**, requires that the least inefficient equilibrium Pareto dominates any other equilibrium. That is, WPNI might implement several equilibria (and all of them might be inefficient), but the least inefficient equilibrium should be preferred by all the agents to any other equilibrium. Thus, if the Nash equilibrium is a good predictor of the outcome implemented by the mechanism, the least inefficient equilibrium stands out as a reasonable selection.[2]

PNI and WPNI suggest implementation with equilibrium selection rather than arbitrarily choosing the best or the worst equilibria. Either the efficient equilibrium is implemented under PNI, or the least inefficient is implemented under WPNI. Therefore the price of stability, which uses the best equilibrium as a benchmark, is very well justified under PNI and WPNI.

Minimal information setting

We study the problem of designing mechanisms when the information available to the designer is minimal. Specifically, we focus on the case where the mechanism splits the total cost of the network formed by using only the *costs of the paths demanded by the agents* and

[1] See for instance [34] for a comparison of three cost-sharing mechanisms using the price of anarchy. See [21] for a comparison of two mechanisms in the problem of commons using the worst-absolute surplus loss.

[2] Apart from the property of being immune to unilateral deviation, the Pareto optimal Nash equilibrium is also immune to deviation by the grand coalition. Also, pre-play communication leads to the payoff-dominant Pareto optimal Nash equilibrium in many games. See for instance [4, 8, 24].

the *total cost of the network formed.*

This setting has multiple applications. For instance, consider the network of roads in a state, district or country to be financed by the users of the roads. The procurement of information on the exact paths used by drivers needs the compulsory installment of GPS (global positioning system) in all vehicles and the data to be stored and updated by a central taxing authority. Because of privacy issues this may not be possible politically (see, for example, [19]). However, a tax based on the number of miles driven can be implemented without raising such privacy concerns. Road maintenance taxes based on the miles driven by every user have been used in pilot programs in Oregon since January 2009, and other states such as Ohio, Pennsylvania, Colorado, Florida, Rhode Island, Minnesota and Texas are considering them (see [14, 15, 16, 17, 18, 19]). This kind of environment requires mechanisms where the input is the total cost of the paths used by the agents rather than the paths themselves.

Moreover, in spite of the information on the paths being available, it may sometimes be desirable to use just the total costs of the paths rather than the paths themselves. Consider, for instance, a big or highly dynamic network structure, where agents join and leave the network continuously. It may be impractical to change the formulae of our mechanism every time the network changes. One such example is sharing the cost of a telephone network or the Internet where the agreement is generally monthly but there are agents entering and leaving the network continuously. Notice that charging the same amount for long distance calls makes sense irrespective of the number of users who share the edges.[3] Alternative examples include the fare charged by a taxi (which usually depend only on the distance driven) or the division of a joint electricity bill in condominiums. There are normative concerns too for penalizing agents who may not be responsible for the fact that their links are not shared by a lot of users. Examples are electricity/water supply or postal service to remote villages. There is a reasonable case against charging higher price for these services to the poor villagers living in a small village on the top of a mountain.

This setup has a natural resemblance to the classic rationing problem (also referred to in the literature as a bankruptcy, taxation or claims problem), where a given amount of a resource (e.g., money) must be divided among beneficiaries with unequal claims on the resource.

Overview of the results

Theorem 1 characterizes the class of mechanisms that satisfy PNI. The mechanisms are monotonic in the total cost and do not depend on the demands of the agents. The average cost mechanism (AC) ([36] [21]); which divides the total cost of the network equally among its participants (Theorem 2), is the only symmetric mechanism in this class. These mechanisms are also characterized under Strong Nash Implementation, which requires the efficient equilibrium to be a Strong Nash equilibrium.

[3]The choice of path is not a strategy for the telephone user and thus the setting is not exactly the same. But the cost-sharing method has a similar motivation; namely, it is simpler than charging every caller differently based on the path used.

The main downside of AC and the above variations is that they do not meet individual rationality (IR, also referred to in the literature as voluntary participation): agents demanding cheap links may pay more than the cost of their demands; thus they may subsidize agents who demand expensive links. We provide a class of mechanisms that meet both IR and WPNI. The egalitarian mechanism (EG, [42]), a rule reminiscent of AC that meets IR, also satisfies WPNI. Theorem 3 introduces a new class of rules that are non-symmetric variations of EG. Such rules are the only rules that meet WPNI.

We show that EG has a PoS equal to $H(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$, where k is the number of agents in the network. EG is also an optimum across all rules meeting IR under the PoS measure.

Related literature

The literature on connection networks has been mainly focused on the performance of the Shapley mechanism.[4] For example, [6] studies the equilibrium behavior of separable mechanisms. The PoS of separable mechanisms with a linear cost-sharing function is at least $H(k)$ (which is $O(\log k)$), where k is the number of agents [6]. $H(k)$ is the upper bound on PoS(Sh) in general graphs, [2]. This upper bound is achieved in directed graphs. If the graph is undirected, PoS(Sh) is lower than $H(k)$. [11] finds a new upper bound of $O(\log \log k)$ when the graph is single source and there are no Steiner nodes. [27] finds a new upper bound of $O(\log k / \log \log k)$ for single source networks when Steiner nodes are allowed. [6] shows that the upper bound in a two player case with single source is $\frac{4}{3}$. [26] finds out that $\frac{4}{3}$ is also the upper bound in a general multi-commodity case. [10] investigates the conditions on network topologies that admit a strong equilibrium under Sh and finds the upper bound on Strong Price of Anarchy ([1]) under Sh to be $H(k)$. [13] considers a problem similar to ours where the designer's objective is to implement the minimum cost spanning tree but the private information about the link costs is not known to the designer. They characterize the set of cost-sharing rules under which true revelation of link costs is a NE.

The paper also connects to the literature on the rationing problem (also refereed as bankruptcy, taxation or claims problem). This literature was started by [38, 3] and nicely surveyed by [37, 43]. The class of asymmetric parametric methods plays a key role in Theorem 3. [45] characterizes the class of symmetric parametric methods by consistency in the population of agents, continuity and symmetry. This class is extended to richer settings by [23]. The axiom of consistency, which is the key component of the parametric methods, has been extensively explored in the literature. See [7, 22, 44] for characterizations of the rules that are consistent. [33, 5, 43, 37, 46] provide important collections of consistent rules.

[4]The Shapley mechanism, even though it looks like a natural mechanism in this setting, fails basic tests such as efficiency, symmetry at equilibrium and continuity. It also does not satisfy minimal information, since the cost share of an agent depends on the number of users of his demanded links.

2 The Model

We denote the set of agents by $\bar{K} = \{1, 2, \ldots, k\}$. A network cost-sharing problem is a tuple $N =< G, K >$, where $G = (V, E)$ is a network that is directed or undirected such that each edge $e \in E$ has a non-negative cost c_e. $K = \{\{s_1, t_1\}, \{s_2, t_2\}, \ldots, \{s_k, t_k\}\}$, where $\{s_i, t_i\} \in 2^V$ for all $i \in \bar{K}$, is the set of sources and sinks that agents want to connect. When there is no confusion, we also denote $K = \bar{K}$ as the set of agents. Let the set of all graphs be $\mathbf{G}$, and the set of all network cost-sharing problems be denoted by $\mathbf{N}$.

Given a problem $N \in \mathbf{N}$, a strategy for agent i is a path $P_i \subseteq E$ that connects s_i to t_i. Let the set of paths connecting s_i to t_i be $\Pi_i(N)$. Let $\Pi(N) \equiv \underset{i \in K}{\times} \Pi_i(N)$ be the set of strategy profiles of all agents in network N. $P = \{P_i\}_{i=1}^k \in \Pi(N)$ will be used to denote a strategy profile of the agents. When there is no confusion we denote $\Pi_i(N)$ and $\Pi(N)$ simply as Π_i and Π respectively. Let $G_P = (V, \underset{i \in \bar{K}}{\cup} P_i)$ be the network formed by the choice of paths by different agents. Let $C(P) = \sum_{e \in G_P} c_e$ be the cost of the graph formed by strategies P.

Let $\mathcal{N} = \{(P, N) | P \in \Pi(N), N \in \mathbf{N}\}$ the union of all problems with their respective strategies.

Definition 1 *A cost-sharing mechanism is a continuous*[5] *mapping $\varphi : \mathcal{N} \to \mathbb{R}_+^k$ such that*

$$\sum_{i \in K} \varphi_i(P, N) = C(P) \text{ for all } (P, N) \in \mathcal{N}.$$

A cost-sharing mechanism assigns non-negative cost-shares to the users of the network based on their demands such that the total cost of the network formed is exactly collected.

Continuity in the mechanism, which at first looks harmless, plays a key role in the proofs. Continuity captures the fact that small perturbations on the demand or cost of the network should not change the total allocation of the cost.

Example 1
- *The Shapley mechanism, Sh, divides the cost of every link equally across its users, that is, $Sh_i(P, N) = \sum_{e \in P_i} \frac{c_e}{U(e,P)}$ for all $i \in \bar{K}$, where $U(e, P)$ is the number of users of link e in the strategy profile P.*
- *The proportional to the stand-alone mechanism, η^{pr}, divides the cost of the network in proportion to every user's stand-alone cost. That is, $\eta_i^{pr}(P, N) = \frac{SA_i(N)}{SA_1(N)+\cdots+SA_k(N)} C(P)$ for all $i \in \bar{K}$, where $SA_i(N) = \min_{P_i \in \Pi(N)} C(P_i)$ is the stand-alone of agent i in network N.*
- *The Average Cost mechanism, AC, divides the cost of the network formed equally across all users. That is $AC_i(P, N) = \frac{C(P)}{k}$ for all $i \in \bar{K}$.*

The Shapley mechanism is a separable mechanism; that is, it divides the cost of every link only across its users and adds those costs for all links in the network formed. Alternative separable mechanisms can be constructed by considering different cost-sharing rules for

[5] Continuous with the Euclidean distance as a function of the costs in the network.

the links, for instance, by giving priority across all users. Nevertheless, Sh is the optimal mechanism (using the price of stability measure; see below) across all separable mechanisms ([6]). Sh can be computed in polynomial time.

On the other hand, η^{pr} divides the cost of the network in proportion to the stand-alone of the agents. Since the stand-alone of every agent has to be computed for every network, this mechanism uses the full information of the network.

AC divides the cost of the network formed equally across the users of the network. It is the most egalitarian rule, reminiscent of the classic head tax rule, where the size of agents' demands is not relevant; only the size of the total cost of the network formed is relevant. AC uses less information than Sh or η^{pr}, since only the total cost of the network formed and the number of agents are needed to compute the cost-sharing allocation. There is no need to know the stand-alone of the agents, or the users of certain links. As a result, its computation complexity is minimal.

To contrast the allocation of the three mechanisms, consider the network in figure 1, where $c_1 = 2$, $c_2 = 1$, $c_3 = 1$. Assume that the demand of agent 1 is st_2t_1 and the demand of agent 2 is st_2. The Shapley mechanism splits the cost of link st_2 equally among agents, therefore it allocates payments $Sh_1 = \frac{1}{2} + 1$ and $Sh_2 = \frac{1}{2}$. The stand-alone cost of agent 1 equals to 2, and the stand-alone cost of agent 2 equals to 1. Therefore, the proportional to the stand-alone mechanism allocates payments: $\eta_1^{pr} = \frac{2}{3}2 = \frac{4}{3}$ and $\eta_1^{pr} = \frac{1}{3}2 = \frac{2}{3}$. Finally, the average cost mechanism splits the cost equally, therefore $AC_1 = AC_2 = 1$.

Definition 2 *A cost-sharing mechanism φ uses minimal information if for any two problems $N =< G, K >$ and $N' =< G', K' >$ and strategies $P \in \Pi(N)$ and $P' \in \Pi(N')$ such that $C(P_i) = C(P_i')$ for all $i \in \bar{K}$ and $C(P) = C(P')$: $\varphi(P, N) = \varphi(P', N')$.*

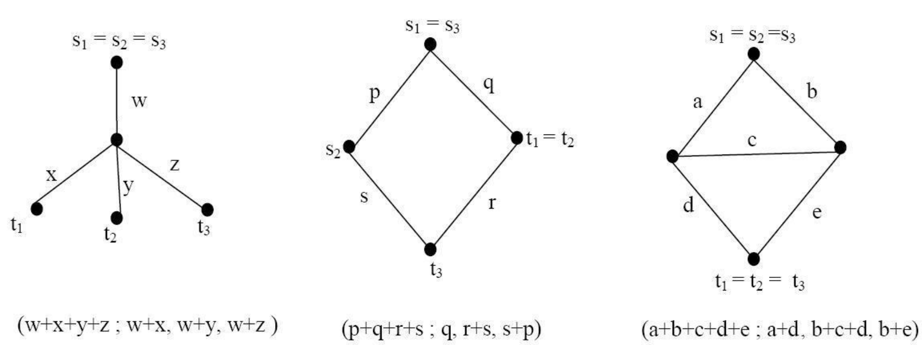

Figure 2: Equivalence in cost-shares under minimal information.

Minimal information captures the mechanisms that depend only on the cost of the network formed and the cost of the agents' demands. For instance, in figure 2, we represent three different networks formed by the demands of three agents. The first coordinate represents the cost of the network formed. The second, third and fourth coordinates represent the cost of the demands of agents 1, 2 and 3 respectively. A cost sharing mechanism that uses minimal information would allocate the same payments to the agents in all three networks if $w+x+y+z = p+q+r+s = a+b+c+d+e$, $w+x = q = a+d$, $w+y = r+s = b+c+d$ and $w+z = s+p = b+e$.

Neither Sh nor η^{pr} use minimal information. On the other hand, AC uses only the total cost of the network formed and the number of users; thus, it uses minimal information. More complex mechanisms that use minimal information are discussed in the following sections.

All the mechanisms studied in this paper use minimal information. Minimal information is always assumed and we do not refer to it when there is no confusion.

Let $\mathcal{S}^k = \{(c; y) \in \mathbb{R}_+ \times \mathbb{R}^k_+ | \max_i y_i \leq c \leq \sum_i y_i\}$.

Lemma 1 *A cost-sharing mechanism φ uses minimal information if and only if there is a continuous function $\xi : \mathcal{S}^k \to \mathbb{R}^k_+$ such that $\sum_i \xi_i(c; y) = c$ for all $(c; y) \in \mathcal{S}^k$, and*

$$\varphi(P, N) = \xi(C(P); C(P_1), \ldots, C(P_k))$$

for all problems $(P, N) \in \mathcal{N}$.

Proof. The sufficiency part is obvious. We prove the necessity only.

First, for any $(c; y) \in \mathcal{S}^k$ we construct the network $\tilde{N}(c; y)$ as follows. Assume without loss of generality that $y_1 \geq y_2 \geq \cdots \geq y_k$. Choose i, $i \in \{1, \ldots k\}$ such that:

$$y_1 + y_2 + \cdots + y_i \leq c < y_1 + y_2 + \cdots + y_{i+1}.$$

Let $\tilde{N}(c; y)$ be a linear network such that every agent has a unique strategy (see figure 3). All agents 1 to i have demand y_i that does not intersect. Agent $i+1$ has demand y_{i+1} such that a segment of length $c - (y_1 + y_2 + \cdots + y_i)$ does not intersect the other agents, and $y_1 + y_2 + \cdots + y_{i+1} - c$ intersects the demand of agent i. Agent j, $j > i+1$ has demand $\tilde{y}_j$ contained in the demand of agent 1.

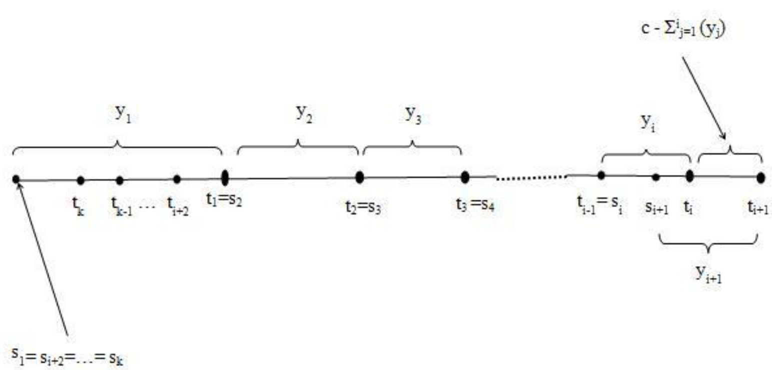

Figure 3: Linear network with all the agents going in the same direction.

Clearly, the unique strategy of agent k in $\tilde{N}(c; y)$ is y_k, and the network formed by all strategies has cost c. Define $\xi : \mathcal{S}^k \to \mathbb{R}^k_+$ as $\xi(c; y) = \varphi(\tilde{N}(c; y))$.

Second, consider any arbitrary network $N =< G, K >$ and a set of demands P. On the one hand, notice that $C(P) \geq C(P_i)$ for every agent i, since $P_i \subseteq P$. On the other hand, notice that $C(P) \leq C(P_1) + \cdots + C(P_k)$, since $P \subseteq P_1 \cup P_2 \cup \cdots \cup P_k$.

Let $y_i = C(P_i)$ and $c = C(P)$. Then $(c; y) \in \mathcal{S}^k$. By minimal information: $\varphi(P, N) = \varphi(\tilde{N}(c; y)) = \xi(c; y)$.

Finally, the continuity of ξ follows from the continuity of φ. ■

Notice that a mechanism that uses minimal information can be expressed as a function ξ that is similar to a rationing method ([43, 37]). Since we work only with mechanisms that use minimal information, we refer without loss of generality to the function ξ as a mechanism. When there is no confusion, the total cost of a path demanded by an agent will be referred to as his demand.

Example 2 *In this example, we see that the network plays a very critical role in the implementation problem of mechanisms that use minimal information. In particular, we see that for the same network, the same total cost and demand may correspond to different equilibria.*

Consider the networks in Figure 2. Assume that $w + x + y + z = a + b + c + d + e$, $w + x = a + d$, $w + y = b + c + d$ *and* $w + z = b + e$. *Thus, any mechanism that uses minimal information allocates the same payments at the given demands.*

The demand profile in Figure 2 (left) will always be equilibrium in any minimal information mechanism. The reason is that there are no other alternatives to the players. However, when we consider the third network in Figure 2 (right), the same demand profile will not be an equilibrium for the AC mechanism for any positive c. Player 1 and 2 have profitable deviations to be from ad and bcd respectively.

Therefore, the equilibria do depend on the network chosen.

2.1 Efficiency and other Desirable Properties

Since we only focus in minimal information mechanisms,[6] all the desirable properties that we require are imposed in the function ξ given by Lemma 1.

Given a problem $N =< G, K >$, we say P^* is an *efficient* graph if $P^* \in \arg\min_{P \in \Pi(N)} C(P)$. That is, P^* is a graph that connects all the agents at a minimal cost. Let $Eff(N)$ be the set of efficient graphs in the problem N.

Given the problem $N =< G, K >$, the mechanism ξ induces the following non-cooperative game $\Gamma^{\xi}(N) \equiv< \bar{K}, \{\Pi_i(N)\}_{i \in \bar{K}}, \{\xi_i\}_{i \in \bar{K}} >$, where the representation of the game is the standard representation of the game in normal form. Namely, $\bar{K} = \{1, \ldots, k\}$ is the set of players, $\Pi_i(N)$ is the strategy space of player i, and ξ_i is the (negative of) payoff function of player i that maps a strategy profile to real numbers.

[6] Section 5 is the only exception, where we see the difficulty of studying mechanisms that do not satisfy minimal information.

P is a *Nash Equilibrium (NE)* of $\Gamma^{\xi}(N)$ if $P_i \in \underset{\acute{P}_i \in \Pi_i(N)}{\arg\min} \xi_i(\acute{P}_i, P_{-i})$ for all i.

Let $NE(\Gamma^{\xi}(N)) \equiv \{P \in \Pi(N) | P \text{ is a Nash Equilibrium of } \Gamma^{\xi}(N)\}$ be the set of Nash equilibria of the game $\Gamma^{\xi}(N)$.

Since every agent in the game $\Gamma^{\xi}(N)$ has a finite number of strategies, the game has a finite number of equilibria.

We say that ξ (weakly) *implements* P, if $P \in NE(\Gamma^{\xi}(N))$.

Definition 3 *The mechanism ξ is efficient (EFF) if it implements an efficient graph for any problem N, that is $P^* \in NE(\Gamma^{\xi}(N))$ for some efficient graph P^*.*

The definition of efficiency just requires an efficient graph to be selected as a Nash equilibrium. This does not preclude other equilibria from being selected.

Notice that AC is efficient. Indeed, at any efficient strategy profile P^*, every agent is paying $\frac{C(P^*)}{k}$. If an agent i deviates from P^* to $\tilde{P}_i$, then he will pay $\frac{C(P_i, P^*_{-i})}{k}$. Clearly, $\frac{C(P_i, P^*_{-i})}{k} \geq \frac{C(P^*)}{k}$ by the optimality of P^*.

Section 4 discusses a variety of mechanisms that are not efficient.

For the vectors $z, \tilde{z} \in \mathbb{R}^m$, we say $z \leq \tilde{z}$ if $z_i \leq \tilde{z}_i$ for all i.

Definition 4 *The mechanism ξ Pareto Nash Implements (PNI) an efficient graph if for any problem N, it implements an efficient graph and that graph Pareto dominates any other equilibrium. That is, for any problem N:*

- *There is an efficient graph P^* such that $P^* \in NE(\Gamma^{\xi}(N))$, and*
- *For any other $P \in NE(\Gamma^{\xi}(N)) : \xi(P^*) \leq \xi(P)$.*

PNI is a very robust property that guarantees that the efficient allocation is selected even when a multiplicity of equilibria arise. In the case of a multiplicity of equilibria, PNI guarantees that **all** agents would prefer the efficient graph to any other equilibrium. Hence, under multiplicity of equilibria, it serves as a selection rule.

In particular, this guarantees that whenever there is a multiplicity of equilibria such that agent i prefers equilibrium P^i to P^j, and agent j prefers equilibrium P^j to P^i, there should exist another equilibrium P^* (the efficient equilibrium) such that agent i prefers equilibrium P^* to P^i and agent j also prefers equilibrium P^* to P^j.

The AC mechanism is also PNI. Indeed, at the efficient graph P^*, this equilibrium would Pareto dominate any other equilibrium $\tilde{P}$ since $\frac{C(P^*)}{k} \leq \frac{C(\tilde{P})}{k}$.

Another point in favor of AC (and its asymmetric variations discussed below) is that it generates an ordinal potential game on the set of players where the potential function is equal to the total cost. Since the vast family of decentralized learning/tatonnement mechanisms converge to a Nash equilibrium in a potential game,[7] they will also do so in the AC mechanism. Moreover, with the presence of a non-binding coordinator (who knows the optimal path in advance), the agents can easily converge to the best Nash equilibrium.

[7]See [30, 31] for convergence of fictitious play and best reply dynamics. See [41] for more general dynamics.

Definition 5 *The mechanism ξ Strongly Nash Implements (SNI) an efficient graph if for any problem N it implements an efficient graph in a strong Nash equilibrium. That is, for any problem N,*

- *There is an efficient graph P^* such that $P^* \in NE(\Gamma^{\xi}(N))$, and*
- *For any group of agents $S \subset \{1, \dots, k\}$, and $P \in \Pi(N)$ such that $P_{-S} = P^*_{-S}$, if $\xi_i(P) < \xi_i(P^*)$ for some $i \in S$, then $\xi_j(P) > \xi_j(P^*)$ for some $j \in S$.*

Under SNI there is no group of agents that can coordinate paths and weakly improve all of them, and at least one agent in the group strictly improves. This is similar to the strong Nash equilibrium and to the literature on group strategyproofness ([20, 32]).

The AC mechanism is also SNI. Indeed, at any deviation $\tilde{P}_S$ of the group of agent S from the efficient graph P^*, it should be that $\frac{C(P^*)}{k} \leq \frac{C(\tilde{P}_S, P^*_{N \setminus S})}{k}$ for all $i \in S$. Hence no agent in S would strictly improve by deviating.

Definition 6
- *The mechanism ξ that uses minimal information is demand monotonic (DM) if for all feasible problems $(c; y), (c; \tilde{y}) \in \mathcal{S}^k$ such that $y_{-i} = \tilde{y}_{-i}$ and $y_i < \tilde{y}_i$: $\xi_i(c; y) \leq \xi_i(c; \tilde{y})$.*
- *The mechanism ξ that uses minimal information is strongly demand monotonic (SDM) if for all feasible problems $(c; y), (c; \tilde{y}) \in \mathcal{S}^k$ such that $y_{-i} = \tilde{y}_{-i}$ and $y_i < \tilde{y}_i$: $\xi_{-i}(c; y) \geq \xi_{-i}(c; \tilde{y})$.*

Demand monotonicity is a weak property that requires that whenever the demand of an agent increases, everything else fixed, his payment should not decrease. Notice that does not preclude the possibility that the payment of other agents would change. Under SDM, the increase in the demand of one agent does not increase the payment of other agents. In particular, notice that SDM implies DM since all of the agents' payments have to add up to a constant.

AC is clearly strongly demand monotonic since $AC(c; y) = AC(c; \tilde{y})$. Thus an increase in the demand of one agent does not change the payments of the other agents.

3 Implementing the Efficient Equilibrium

We now turn to the first main result of the paper. We characterize the mechanisms that meet the efficiency properties discussed above.

Theorem 1 *Assume there are three or more agents, then the following statements are equivalent for a mechanism ξ that uses minimal information:*

1. *ξ is EFF and SDM.*
2. *ξ PNI the efficient graph.*

3. *ξ SNI the efficient graph.*

4. *There is a monotonic function $f : \mathbb{R}_+ \to \mathbb{R}^k_+$ such that $\sum_i f_i(c) = c$ and for all feasible problems $(c; y)$, $\xi(c; y) = f(c)$.*

The mechanisms characterized by Theorem 1 are demand independent; that is, the cost-share of every agent does not depend on whether the agents are demanding cheap or expensive links. Instead, they depend only on the total cost of the network formed. The average cost mechanism, generated by $f(c) = (\frac{c}{k}, \ldots, \frac{c}{k})$, is the only mechanism in this class that treats equal agents equally.

Notice efficiency alone is not sufficient to characterize the above mechanisms. Indeed, consider the mechanism

$$\tilde{\xi}(c; y) = (\min\{y_3, \frac{c}{k}\}, \frac{2c}{k} - \min\{y_3, \frac{c}{k}\}, \frac{c}{k}, \ldots, \frac{c}{k}).$$

First, notice that $\tilde{\xi}$ implements the efficient graph because at the efficient graph agents $\{3, \ldots, k\}$ do not have the incentive to deviate since by doing so their payment is going to increase. On the other hand, agents $\{1, 2\}$ do not have any incentive to deviate from the efficient equilibrium since the functions $\min\{y_3, \frac{c}{k}\}$ and $\frac{2c}{k} - \min\{y_3, \frac{c}{k}\}$ are weakly monotonic in the total cost of the network and do not depend on their report.

$\tilde{\xi}$ is also an example of a mechanism that is not SNI, but agents cannot *strictly* improve by coordinating.

Minimal information is crucial to get this result. The proportional to the stand-alone mechanism η^{pr} PNI and SNI the efficient graph. However, η^{pr} does not use minimal information.

3.1 Efficient mechanisms for two agents

The example above shows that for three or more agents, EFF is not enough to characterize the demand-independent rules. On the other hand, this property is enough when there are two agents. The property is an immediate consequence of a Separability Lemma described below.

Proposition 1 *Assume there are two agents, $K = \{1, 2\}$. A mechanism that uses minimal information is efficient if and only if there is a monotonic function $f : \mathbb{R}_+ \to \mathbb{R}^2_+$ such that $f_1(c) + f_2(c) = c$ and for all feasible problems $(c; y)$, $\xi(c; y) = f(c)$.*

3.2 Equal treatment of equals

Definition 7 *The mechanism ξ that uses minimal information satisfies equal treatment of equals (ETE) if for all agents i, j and $(c; y) \in \mathcal{S}^k$ such that $y_i = y_j$: $\xi_i(c; y) = \xi_j(c; y)$.*

ETE is the standard property of equal responsibility for the cost of the good. Equal agents with the same demand should be allocated the same cost. There is a large class of solutions that meet ETE. We describe in section 4 alternative rules that meet ETE, such as the Proportional and Egalitarian solution.

Theorem 2 *A mechanism that uses minimal information is EFF and ETE if and only if it is AC.*

Notice that this statement is not directly implied by Theorem 1, since we do not need SDM.

4 Individually Rational Mechanisms

Definition 8 *A mechanism ξ that uses minimal information is individually rational (IR) if for all $(c; y) \in \mathcal{S}^k : \xi_i(c; y) \leq y_i$ for all i.*

Individually rational mechanisms rule out cross-subsidies; that is, no agent pays more than the cost of his demand.

Notice that neither *AC* nor any mechanism discussed in Theorem 1 meet individual rationality. Therefore, the traditional incompatibility of strategy-proofness, efficiency, budget-balance and individual rationality ([12]) also holds in this problem.

This incompatibility holds only because we consider mechanisms that use minimal information. If we remove this constraint, there is a large class of mechanisms that always implement the efficient network and at the same time meet individual rationality. For instance, consider the proportional to the stand-alone mechanism η^{pr} discussed above. η^{pr} is individually rational because no agent pays more than his stand-alone, which in turn is less than his demand. On the other hand, η^{pr} implements the efficient allocation because the cost-share of every agent is in proportion to the cost of the network; therefore, any deviation from the efficient graph that increases the total cost of the network formed would increase the cost-share of all the agents.

On the other hand, there is a large class of individually rational mechanisms that use minimal information: most of the mechanisms discussed in the rationing/bankruptcy literature meet IR; see, for instance, [43, 37].

A class of rationing mechanisms that is especially compelling is the class of asymmetric parametric methods.

Definition 9 *For every agent i, consider $F_i : [0, \Lambda] \times \mathbb{R}_+ \to \mathbb{R}_+$, continuous in both variables, non-decreasing in the first variable and such that $F_i(\lambda, 0) = 0$ and $F_i(\Lambda, z) = z$ for all λ and z. A parametric rationing mechanism is defined as*

$$\varphi_i(c, y) = F_i(\lambda^*, y_i) \text{ where } \lambda^* \text{ solves } \sum_{i \in \bar{K}} F_i(\lambda^*, y_i) = c.$$

The class of asymmetric parametric methods is very rich; it contains almost any rationing method discussed in the literature.[8] In particular, it contains the two basic rationing methods: the proportional and egalitarian mechanism. The proportional mechanism, PR, divides the cost of the agents in proportion to their demands:

$$PR_i(c; y) = \frac{y_i}{y_1 + \cdots + y_k} c.$$

On the other hand, the egalitarian mechanism, EG, divides the cost equally across the agents subject to no agent paying more than his demand:

$$EG_i(c; y) = \min\{y_i, \lambda^*\} \text{ where } \lambda^* \text{ solves } \textstyle\sum_i \min\{y_i, \lambda^*\} = c.$$

The parametric description of these two methods is given by:

Proportional: $F_i(\lambda,z) = \lambda z, \Lambda = 1;$
Egalitarian: $F_i(\lambda, z) = min\{\lambda, z\}, \Lambda = \infty;$

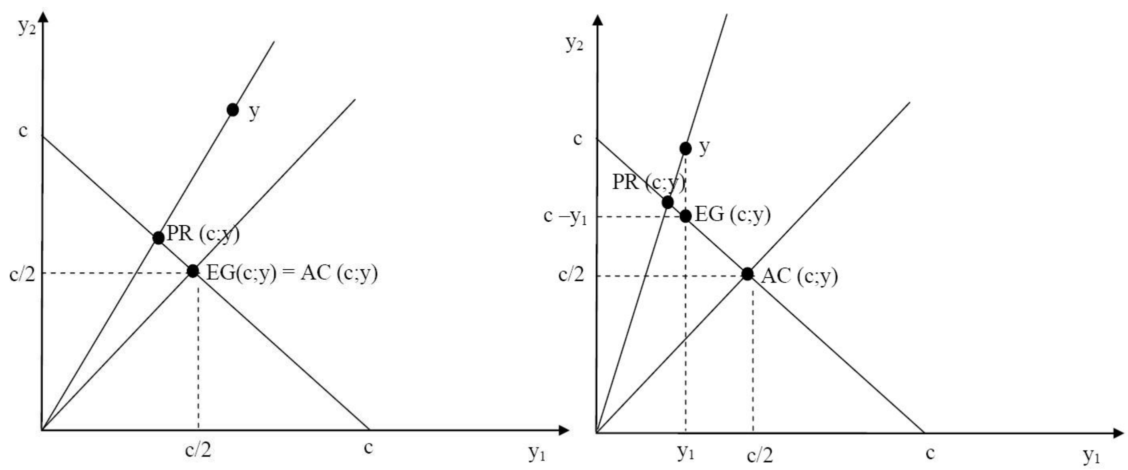

Figure 4: Contrast between AC, EG and PR for two agents.

Figure 4 illustrates the allocation of payments for AC and EG at the problem $(c; y_1, y_2)$. In both figures, AC would allocate a payment equal to $\frac{c}{2}$ to every agent. In figure 4 left, both agents have a demand above the average cost $\frac{c}{2}$. Therefore, EG coincides with AC and allocates a payment of $\frac{c}{2}$ to every agent. On the other hand, in figure 4 right, agent 1 demands less than the average cost $\frac{c}{2}$. Therefore, his payment under EG would be equal to his demand y_1, whereas agent 2 would pay the difference to cover the cost $c - y_1$. Proportional rule just allocates the point of intersection of the simplex with the line joining the origin and the demand vector.

[8]See [45] for a characterization of the symmetric parametric methods; see [37] for a more detailed description of the methods.

We now introduce a class of mechanisms that generalize the egalitarian mechanism. These mechanisms, which resemble a fixed path method, are briefly introduced and discussed in section 1.8 of [37]. To illustrate the class of rules, consider a non-decreasing function $f_i : [0, \Lambda] \rightarrow \mathbb{R}_+$ such that $f_i(0) = 0$ and $f_i(\Lambda) = \infty$, for every agent $i \in \{1, \ldots, k\}$. Given the demands of the agents $(y_1, \ldots, y_k)$ and a cost of the network c, the cost-share of agent i is given by:

$$EG_i^{f_1,f_2,\ldots,f_k}(c; y_1, y_2, \ldots, y_k) = \min\{f_i(\lambda^*), y_i\},$$

where λ^* solves $\sum_{i=1}^k \min\{f_i(\lambda^*), y_i\} = c$.

Notice the mechanism $EG_i^{f_1,f_2,\ldots,f_k}$ clearly meets IR since

$$EG_i^{f_1,f_2,\ldots,f_k}(c; y_1, y_2, \ldots, y_k) \leq y_i.$$

The mechanism $EG_i^{f_1,f_2,\ldots,f_k}$ will be called an *asymmetric egalitarian mechanism (AEM).*

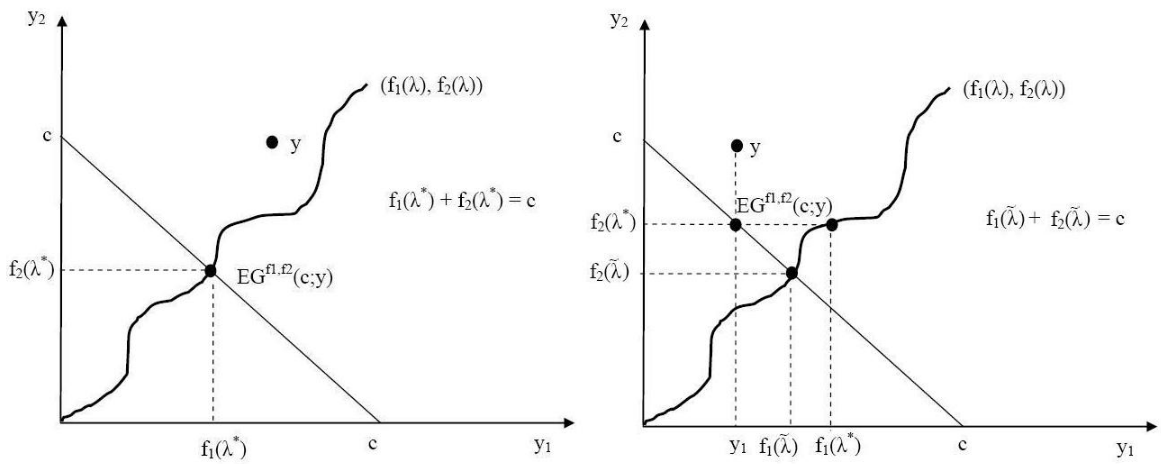

Figure 5: The asymmetric egalitarian mechanisms for two agents.

Figure 5 illustrates the allocation of payments for EG^{f_1,f_2} at the problem $(c; y_1, y_2)$. The path used to compute the payments, $\{(f_1(\lambda), f_2(\lambda))|\lambda \geq 0\}$, is generated by the functions $f_1(\lambda)$ and $f_2(\lambda)$. In figure 5 left, the mechanism would allocate payments equal to $(f_1(\lambda^*), f_2(\lambda^*))$ because $y_1 \geq f_1(\lambda^*)$ and $y_2 \geq f_2(\lambda^*)$. On the other hand, in figure 5 right, the mechanism would allocate payments equal to $(y_1, c - y_1)$ because $y_1 < f_1(\tilde{\lambda})$.

The egalitarian mechanism is constructed by picking functions $f_1 = f_2 = \cdots = f_k$. The path generated by these functions is the identity line $\{(\lambda, \lambda, \ldots, \lambda) \in \mathbb{R}^k|\lambda \geq 0\}$.

Alternatively, the weighted egalitarian methods are constructed when $f_i(\lambda) = w_i\lambda$ for all i, for a given set of constants $w_1, \ldots, w_k$. The path generated by such functions is the ray $\{\lambda(w_1, w_2, \ldots, w_k)|\lambda \geq 0\}$

Contrary to the traditional analysis of this problem, the games induced by the asymmetric egalitarian mechanisms are not potential games; see section 11.2 for an example illustrating that. Therefore, the previous potential techniques used in the analysis of this problem do not work anymore. In general, a mechanism induced by a rationing method could not have a pure strategy Nash equilibrium. Nevertheless, we show below that AEM and proportional mechanisms always have pure strategy Nash equilibria and provide algorithms to compute them.

Lemma 2 *The proportional and the asymmetric egalitarian mechanisms always have a pure strategy Nash equilibrium.*

Proof. We prove the Lemma for an AEM mechanism. The proof for the proportional mechanism is written in the appendix.

Consider the asymmetric egalitarian mechanism φ generated by the functions $f^1, \ldots, f^k$. Let S_i be the stand-alone of agent i, and let $s^i = C(S^i)$ be its cost.

We show that the best reply tatonnement starting from the profile of demands $(S^1, S^2, \ldots, S^k)$ converges to a Nash equilibrium.

Let $g^i(x) = (f^i)^{-1}(x)$ be the inverse of f^i. Assume without loss of generality that $g^1(s^1) \leq g^2(s^2) \leq \cdots \leq g^k(s^k)$.

Let $C(S^1, \ldots, S^k)$, and let λ^* such that $\sum_i \min\{f^i(\lambda^*), s^i\} = C(S^1, \ldots, S^k)$.

Let m such that:

$$g^1(s^1) \leq \cdots \leq g^{m-1}(s^{m-1}) \leq \lambda^* < g^m(s^m) \leq \cdots \leq g^k(s^k).$$

First, notice that $\varphi_i(s_1, \ldots, s_k) = s^i$ if $i < m$, and $\varphi_i(s_1, \ldots, s_k) = f^i(\lambda^*)$ if $i \geq m$.

Second, notice that at the best reply of any agent, the variable λ^* should decrease.

To see this, consider agent j and assume that his best reply is the path Y^j with cost $C(Y^j) = y^j$.

Case 1. If $j < m$, then his cost-share is $\varphi_j(s_1, \ldots, s_k) = s^j$. At his best reply Y^j, $y^j \geq s^j$. The only way to decrease his cost-share is by moving to $\tilde{\lambda}$ such that $f^j(\tilde{\lambda}) < s^j$. Since $g^j(s^j) \leq \lambda^*$, then $s^j \leq f^j(\lambda^*)$. Hence $\tilde{\lambda} < \lambda^*$.

Case 2. f $j \geq m$ then $\varphi_j(s_1, \ldots, s_k) = f^j(\lambda^*) < s^j$. At his best reply Y^j, $C(Y^j) \geq s^j$. Therefore, the only way to decrease his payment is by decreasing λ^*, since he cannot decrease his demand below s^j.

Finally, notice we can replicate cases 1 and 2 above for the new profile (Y_j, S_{-j}). Indeed, notice $\varphi_i(y_j, s_{-j}) = s^i$, or $\varphi_i(y_j, s_{-j}) = f^i(\tilde{\lambda})$. Similar to case 1, an agent paying his stand-alone, that is $\varphi_i(y_j, s_{-j}) = s^i$, will be demanding his stand-alone. Thus, his only profitable deviation will be to decrease $\tilde{\lambda}$ to $\bar{\lambda}$, $\bar{\lambda} < \tilde{\lambda}$, such that $f^i(\bar{\lambda}) < s^i$, thus, decreasing the total cost of the network.

On the other hand, if $\varphi_i(y_j, s_{-j}) = f^i(\tilde{\lambda})$ then $f^i(\tilde{\lambda}) \leq s^i$. Similar to case 2, an increase in their demand is only profitable if $\tilde{\lambda}$ decreases, and so does the total cost of the network.

Since at any step the value λ^* decreases, it is bounded and there is a finite number of strategies, λ^* converges in a finite number of iterations. The limit profile is a Nash equilibrium. ■

4.1 Weakly Pareto Nash Implementation

There are no individually rational mechanisms that meet PNI. **Weakly Pareto Nash Implementation (WPNI)** requires that the least inefficient equilibrium Pareto dominate any other equilibrium. That is, WPNI might implement several equilibria (and all of them might be inefficient), but the least inefficient equilibrium should be preferred by all the agents to any other equilibrium. Clearly if a mechanism satisfies PNI, then it satisfies WPNI.

Definition 10 *A mechanisms ξ satisfies WPNI if:*

i. For any problem N, ξ has at least one Nash equilibrium.

ii. Let P^* be the the equilibrium profile with the minimal cost, that is $C(P^*) \leq C(\tilde{P})$ for any other equilibrium $\tilde{P}$. Then, $\xi(P^*) \leq \xi(\tilde{P})$.

WPNI serves an equilibrium selection rule. If the Nash equilibrium is a good predictor of the outcome implemented by the mechanism, then the least inefficient equilibrium stands out as a selection since all the agents prefer it.

Theorem 3 *An asymmetric parametric mechanism meets WPNI if and only if it is an asymmetric egalitarian mechanism.*

All the mechanisms discussed in this section are inefficient. The measure below will serve as a selection criteria for different mechanisms.

Definition 11 *The price of stability (PoS) of the mechanism ξ equals:*

$$\max_{N \in \mathcal{N}, P^* \in Eff(N)} \left\{ \frac{\min_{P \in NE(\Gamma^{\xi}(N))} C(P)}{C(P^*)} \right\}$$

The price of stability, which computes the ratio between the best efficient equilibrium and the efficient outcome, is a compelling measure of the inefficiency generated by WPNI mechanisms since the agents' incentives are aligned to pick the best Nash equilibrium.

Notice that PoS is always greater than or equal to 1. A mechanism is efficient if it has a price of stability equal to 1. The smaller the price of stability, the more efficient is the mechanism.

Corollary 4 *i. EG has the smallest price of stability across all asymmetric parametric mechanisms meeting WPNI. It has a price of stability equal to $H(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$.*

ii. EG minimizes the price of stability across all individually rational mechanisms.

iii. The price of stability of PR is of order k.

Since the Shapley mechanism has a PoS equal to $H(k)$, EG is no more inefficient than the Shapley mechanism. No other individually rational mechanism can be more efficient than EG and Shapley. On the other hand, the traditional proportional mechanism is extremely inefficient; since its price of stability is bounded by k, its maximal loss approaches that in the limit.

4.2 Strong Nash Implementation

Definition 12 *A mechanism ξ strongly Nash implements the equilibrium with the minimal cost (SNIMC) if*

i. For any problem N, ξ has at least one Nash equilibrium.

ii. Let P^* be the the equilibrium profile with the minimal cost, that is $C(P^*) \leq C(\tilde{P})$ for any other equilibrium $\tilde{P}$. Then, P^* is a Strong Nash Equilibrium; that is, if $\xi_S(P_S, P^*_{N\setminus S}) \geq \xi_S(P^*)$ for some demands P_S of the agents in S, then $\xi_S(P_S, P^*_{N\setminus S}) = \xi_S(P^*)$

If a mechanism strongly Nash implements the efficient graph (SNI), then it strongly Nash implements the equilibrium with the minimal cost (SNIMC). The converse is not true, since the equilibrium with the minimal cost might not be efficient. The AEM are such examples.

Proposition 2 *The asymmetric egalitarian mechanisms SNIMC.*

Notice that together with the WPNI property of AEG, this Proposition implies that there is one and only one strong Nash equilibrium under AEG. This comes from the fact that the Nash equilibria are Pareto ranked and thus under any NE other than the cheapest NE, the grand coalition has a profitable deviation. This means that the strong price of anarchy (SPoA) of the EG equals the strong price of stability of EG, and they equal $H(k)$. This earmarks another advantage of EG over Sh, since we know that Sh does not always admit Strong NE [2, 6] and therefore, SPoA does not exist for Sh.

We conjecture that the only mechanisms that SNIMC are the asymmetric egalitarian mechanisms.

5 Mechanisms under full information

The class of mechanisms that use full information and at the same time PNI or SNI the efficient graph is very large. The proportional to the stand-alone mechanism η^{pr} described above belongs to the class. Alternatively, instead of dividing in proportion to the stand-alone, we can divide in equal parts or give priority to agents.

Recall that $SA_i(N) = \min_{P_i \in \Pi(N)} C(P_i)$ is the stand-alone of agent i in problem N. In other words, $SA_i(N)$ is the cost of the shortest (cheapest) path connecting s_i and t_i in N.

Example 3

- *The average-cost up to the stand-alone mechanism, η^{AC}, divides the cost of the network equally subject to no agent paying more than his stand-alone cost. That is, given the stand-alone costs of the agents such that $SA_1(N) \leq SA_2(N) \leq \ldots, \leq SA_k(N)$:*

  $\eta_1^{AC}(P, N) = \min\{SA_1(N), AC(P, N)\}$, *and*

  $\eta_i^{AC}(P, N) = \min\{SA_i(N), [C(P) - \sum_{j=1}^{i-1} \eta_j^{AC}(P, N)]/(k-i+1)\}$, *for* $i = 2, 3, \ldots, k-1$.

  $\eta_k^{AC}(P, N) = \max\{C(P) - \sum_{j=1}^{k-1} \eta_j^{AC}(P, N), 0\}$

- *The priority up to the stand-alone mechanism, $\eta^{\succ}$, divides the cost of the network following a fixed priority $\succ$ of the agents. That is, for the fixed priority of the agents $1 \succ 2 \succ \cdots \succ k$:*

  $\eta_1^{\succ}(P, N) = \min\{SA_1, C(P)\}$, *and*

  $\eta_i^{\succ}(P, N) = \min\{SA_i, C(P) - \sum_{j=1}^{i-1} \eta_j^{\succ}(P, N)\}$, *for* $i = 2, 3, \ldots, k-1$.

  $\eta_k^{\succ}(P, N) = \max\{C(P) - \sum_{j=1}^{k-1} \eta_j^{\succ}(P, N), 0\}$

η^{pr}, η^{AC} and $\eta^{\succ}$ PNI and SNI the efficient graph because the cost share of every agent at the efficient graph is no larger than his cost-share at any other graph. Moreover, these mechanisms also respect the stand-alone property at the efficient equilibrium. That is, at the efficient equilibrium, no agent pays more that his stand-alone cost because $SA_1(N) + \cdots + SA_1(N) \geq C(P^*)$ for any efficient graph P^*.

η^{pr}, η^{AC} and $\eta^{\succ}$ belong to the more general class of mechanisms where the efficient graph Pareto-dominate any other graph.

Example 4 *Consider a mechanism ξ such that for any problem N the efficient graph is a Pareto dominant equilibrium. That is, for every efficient graph P^*, $\xi(P^*) \leq \xi(P)$ for any $P \in \Pi(N)$.*

Clearly, the mechanisms given by example 4 PNI and SNI the efficient graph, since every agent prefer the efficient graph to any other graph in the network. The mechanisms characterized by Theorem 1 also belong in this class.

Neither PNI nor SNI imply that the efficient graph is a Pareto dominant equilibrium. For instance, consider the example given in figure 1 and assume that $c_1 = 1$, $c_2 = 1$ and $c_3 = 2$. The efficient graph requires agent 1 and 2 to connect directly, that is st_1 and st_2. Consider the allocation of payments given by the following cost-sharing method:

	st_2	st_2t_1
st_1	1,1	$\frac{1}{2}$, $\frac{5}{2}$
st_2t_1	$\frac{5}{2}$,$\frac{1}{2}$	2,2

Then, the efficient equilibrium is the unique Nash equilibrium, but it is not a Pareto dominant equilibrium. Similar examples can be given for any normal form game where the efficient equilibrium Pareto dominates any other equilibrium.

The full class of mechanisms that meet WPNI or SNIMC is also very difficult to characterize. For instance, the convex combination[9] between an AEM and a mechanism given by example 4 would also meet WPNI and SNIMC. However, the convex combination between asymmetric egalitarian mechanisms might not satisfy WPNI nor SNIMC.[10]

[9] In the usual way, for every strategy profile.

[10] For instance in the case of two agents, the contest garment method is the average of the two priority methods that give absolute priority to either of the agents, see [37] pp. 18. The priority methods are AEM, however the contest garment method is a Parametric method that is not an AEM. Therefore, by Theorem 3, the contest garment method does not meet WPNI.

Therefore, unless there is a more convenient structure on the mechanisms, such as minimal information, the descriptions of the mechanisms meeting PNI, SNI, WPNI and SNIMC would be very difficult to characterize.

6 Conclusions

This paper provides a new perspective on the problem of cost-sharing in networks. In particular, it provides new concepts of implementation and characterizes the classes of mechanisms that meet them. The average cost mechanism and reminiscent variations are characterized using those properties.

It also provides the first economic justification for the price of stability, a seemingly natural measure in the computer science literature but one not easily embraced in economics. It illustrates the environments where PoS is the correct measure to use. The egalitarian mechanism, an optimal mechanism using this measure, is singled out in the class of WPNI mechanisms. Moreover, EG outperforms the traditional Shapley mechanism on several grounds, including efficiency and information requirements.

7 Proof of Theorems 1 and 2

7.1 Preliminary Lemmas

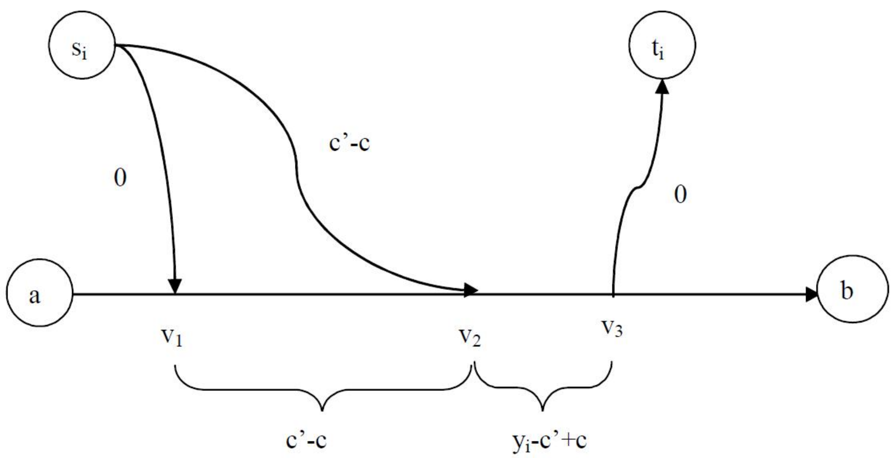

Figure 6: EFF implies cost monotonicity (case 1).

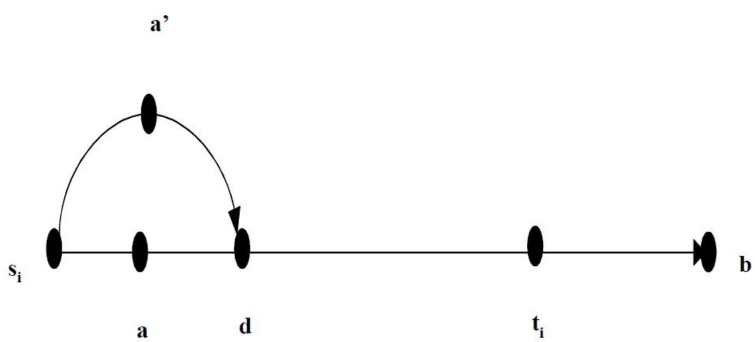

Figure 7: EFF implies cost monotonicity (case 2).

Definition 13 *The mechanism ξ that uses minimal information is monotonic in cost if for all feasible problems $(c; y), (c'; y) \in \mathcal{N}^K$ such that $c < c'$: $\xi(c; y) \leq \xi(c'; y)$.*

Lemma 3 *If the mechanism ξ that uses minimal information is efficient then it is monotonic in total cost.*

Proof.

Consider two feasible problems $(c;y)$ and $(c';y)$, where $c' > c$ and $(c'-c) < \min_{i\in K}\{y_i\}$. Suppose there exists an agent i and an efficient ξ such that $\xi_i(c';y) < \xi_i(c;y)$.

We construct a network that have the two potential profiles $(c;y)$ and $(c';y)$.

Indeed, consider a network where agents $j \neq i$ have just one strategy each, P_j, which costs y_j. Agent i has two strategies P_i and P_i' both of which cost y_i but P_i makes the total cost of the network c, and P_i' makes the total cost go up to c'.

Case 1: $c \leq \sum_{j\neq i} y_j$.

In this case we can have a configuration as shown in figure 6. Here, the demands of agents in $K\backslash\{i\}$ is contained in the interval $a \to b$, which costs c. This is possible since when $c = \sum_{j\neq i} y_j$, we can have $a \to b$ as the concatenation of the demand links of the agents $j \neq i$ .When $c < \sum_{j\neq i} y_j$, we can have the demand links overlapping, e.g., when $\max_{j\neq i}\{y_j\} = c$, then $a \to b$ is the demand link of the biggest demander and all other demands overlap with his. $P_i = s_i \to v_1 \to v_2 \to v_3 \to t_i$ and $P_i' = s_i \to v_2 \to v_3 \to t_i$. All the costly links of P_i are contained in $\{\bigcup_{j\neq i} P_j\}$ whereas there are links of cost $c'-c$ that are not contained in $\{\bigcup_{j\neq i} P_j\}$ under P_i'. Again, this is possible since c' and c are close enough to guarantee that for all i we can have such paths.

Case 2: $\sum_{j\in K} y_j > c > \sum_{j\neq i} y_j$.

In this case we can have a configuration as shown in figure 7. Here, the interval $a \to b$ is the concatenation of the demand links of agents in $K\backslash\{i\}$. Thus $|a \to b| = \sum_{j\neq i} y_j$, $|s_i \to a| = c - \sum_{j\neq i} y_j$, $|a \to d| = c' - c$. $|s_i \to a \to d| = |s_i \to a' \to d| = c' - \sum_{j\neq i} y_j$. $P_i = s_i \to a \to d \to t_i$ and $P_i' = s_i \to a' \to d \to t_i$. Notice that it may be the case that $t_i = b$.

Now clearly in both cases, i will have a profitable deviation from the efficient graph of cost c, thus contradicting the efficiency of ξ. Thus we have shown that efficient ξ must be monotonic in total cost in some open neighborhood of c, for all c. Therefore, we can extend the argument to conclude that ξ must be monotonic in total cost in general. ■

Lemma 4 (Separability Lemma) *If the mechanism ξ that uses minimal information is efficient then* $\implies \xi(C;y) = (\xi_1(C;y_{-1}), \xi_2(C;y_{-2}), \ldots\ldots, \xi_k(C;y_{-k}))$. *That is, any efficient mechanism is separable and assigns the costs-shares to the agents independently of their demand.*

Proof. If we prove that for any feasible problems $(c;y)$ and $(c;\tilde{y}_i, y_{-i})$, any continuous and efficient ξ must have $\xi_i(c;y) = \xi_i(c;\tilde{y}_i, y_{-i})$, then we are done. Consider a feasible problem $(c;y)$. Consider a graph as shown in Figure 8, which generates this problem. The sources and sinks of agents $j \neq i$ lie on the ray $a \to b$ according to the demand profile; i.e., the agent with the highest demand covers most of the span on $a \to b$ and so on. Thus, an agent $j \neq i$ has one strategy that generates the demand y_j. Agent i has two strategies– either connect $s_i - t_i$ through v_1 or through v_2. The demands of agent i when connecting through v_1 and v_2 are $\tilde{y}_i$ and y_i respectively. Now, the total cost when i uses v_1 and v_2 are respectively $c + \epsilon$ and c. Notice that by moving the position of v_2 and arranging the demand links of

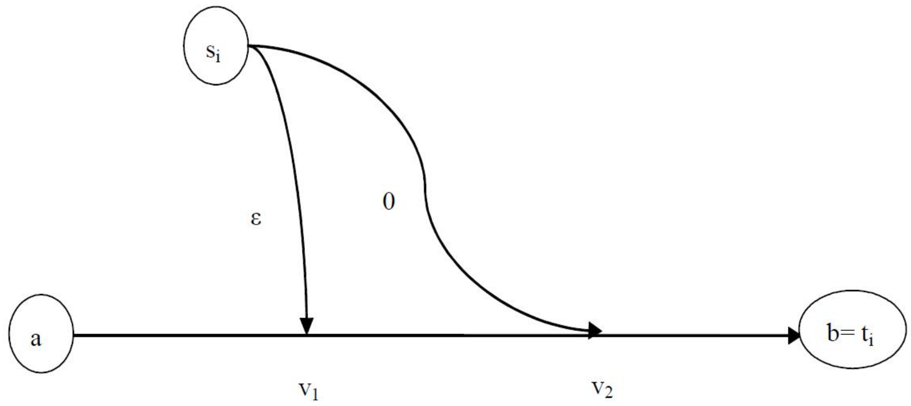

Figure 8: EFF implies separability.

the agents $j \neq i$, we can generate all the feasible problems $(c; y_i, y_{-i})$. Also, by moving the position of v_1 and arranging the demand links of the agents $j \neq i$, we can generate all the feasible problems $(c + \epsilon; y_i, y_{-i})$. Consider an efficient ξ that is continuous. The efficiency of ξ requires the following inequality

$$\xi_i(c; y_i, y_{-i}) \leq \xi_i(c + \epsilon; \tilde{y}_i, y_{-i}) \tag{1}$$

Using continuity we get

$$\xi_i(c; y_i, y_{-i}) \leq \xi_i(c; \tilde{y}_i, y_{-i}) \tag{2}$$

Similarly, switching the position of v_1 and v_2 and using continuity again we get

$$\xi_i(c; y_i, y_{-i}) \geq \xi_i(c; \tilde{y}_i, y_{-i}) \tag{3}$$

Thus, we conclude that $\xi_i(c; y_i, y_{-i}) = \xi_i(c; \tilde{y}_i, y_{-i})$ for all feasible problems $(c; y_i, y_{-i})$ and $(c; \tilde{y}_i, y_{-i})$.

■

7.2 Proof of Proposition 1

Consider a problem $(c; y_1, y_2) \in S^2$.

By Separability Lemma: $\xi_1(c; y_1, y_2) = \xi_1(c; c, y_2)$.

By budget balance: $\xi_2(c; y_1, y_2) = \xi_2(c; c, y_2)$. Thus, $\xi(c; y_1, y_2) = \xi(c; c, y_2)$.

By Separability Lemma: $\xi_2(c; c, y_2) = \xi_2(c; c, c)$.

By budget balance: $\xi_1(c; c, y_2) = \xi_1(c; c, c)$. Thus, $\xi(c; c, y_2) = \xi(c; c, c)$.

Therefore $\xi(c; y_1, y_2) = \xi(c; c, c)$.

Let $f(c) = \xi(c; c, c)$. Since the mechanism is monotonic in the total cost (lemma 3), $f(c)$ is monotonic in the total cost.

7.3 Proof of Theorem 1

7.3.1 1. $\implies$ 4.

Proof.

Consider a continuous ξ that is efficient and strongly monotonic. Consider two arbitrary feasible problems $(c; y)$ and $(c; \tilde{y})$. We will prove that $\xi(c; y) = \xi(c; \tilde{y}) = f(c)$. The monotonicity of f comes from Lemma 3. Let $a = \frac{1}{k}\sum_{i \in K} y_i$ and $\tilde{a} = \frac{1}{k}\sum_{i \in K} \tilde{y}_i$. Assume without loss of generality that $y_1 \leq y_2 \leq y_3 \leq ...... \leq y_k$ and $\tilde{y}_1 \leq \tilde{y}_2 \leq \tilde{y}_3 \leq ...... \leq \tilde{y}_k$.

Step 1: $\xi(c; y) = \xi(c; a, a, ..., a)$ and $\xi(c; \tilde{y}) = \xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$

Proof:

Consider the following problems: $P_0 = (c; y)$, $P_1 = (c; a, y_2, y_3, ...y_k)$, $P_2 = (c; a, a, y_3, y_4, ..., y_k)$, ...,$P_k = (c; a, a, ..., a)$. Notice first that the feasibility of P_0 implies the feasibility of $P_1, P_2, ..., P_k$. This is true because the maximum of the demand profile does not go above y_k in all these problems and the sum of the individual demands is always at least $k * a = \sum_{i \in K} y_i$. Similarly, if we define the counterpart problems $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, ..., \tilde{P}_k$ where $\tilde{P}_i = (c; \tilde{a}, \tilde{a}, ..., \tilde{a}, \tilde{y}_{i+1}, \tilde{y}_{i+2}, ..., \tilde{y}_{k-1}, \tilde{y}_k)$, then again all of them will be feasible.

Now, due to the Separability Lemma, we must have $\xi_1(P_0) = \xi_1(P_1)$. But then strong monotonicity and budget balancedness imply $\xi_{-1}(P_0) = \xi_{-1}(P_1)$. Thus, we have $\xi(P_0) = \xi(P_1)$. Using the same argument, we have $\xi(P_i) = \xi(P_{i+1})$ and $\xi(\tilde{P}_i) = \xi(\tilde{P}_{i+1})$ for all $0 \leq i \leq k-1$. Thus, we have $\xi(P_0) = \xi(P_k)$ and $\xi(\tilde{P}_0) = \xi(\tilde{P}_k)$ as desired.

Step 2: $\xi(c; a, a, ..., a) = \xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$

Proof:

Notice first that the feasibility of $(c; a, a, ..., a)$ & $\xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$ implies that any problem $(c; \hat{a})$ where some of the $\hat{a}_i = a$ and other $\hat{a}_i = \tilde{a}$ is also feasible. Now, the Separability Lemma implies $\xi_1(c; a, \tilde{a}, ..., \tilde{a}) = \xi_1(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$. Now, there can be three cases–$a < \tilde{a}$, $a > \tilde{a}$ or $a = \tilde{a}$. In the first two cases strong monotonicity and budget-balancedness imply $\xi_{-1}(c; a, \tilde{a}, ..., \tilde{a}) = \xi_{-1}(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$ and we get $\xi(c; a, \tilde{a}, ..., \tilde{a}) = \xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$. The third case trivially implies $\xi(c; a, \tilde{a}, ..., \tilde{a}) = \xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$ since it is the same problem so the solution must be the same. Similarly, we get $\xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a}) = \xi(c; a, \tilde{a}, ..., \tilde{a}) = \xi(c; a, a, \tilde{a}, ..., \tilde{a}) = ... = \xi(c; a, a, ..., a)$.

■

7.3.2 2. $\implies$ 1.

Proof.

We know that ξ PNI efficient graph implies that ξ is efficient. We will prove that if ξ PNI the efficient graph then ξ is strongly monotonic. Consider a ξ that PNI the efficient graph and a feasible problem $(c; y)$ and assume without loss of generality that $y_1 < y_2 < ..... < y_k$[11]. Now, consider a graph as shown in figure 9 below.

Here every agent has two strategies– either use the path in the solid graph or use that in the dotted graph. We call the solid graph ** and the dotted graph *. Let * be a small

[11] The case of weak inequality will follow from the assumption of continuity on our method

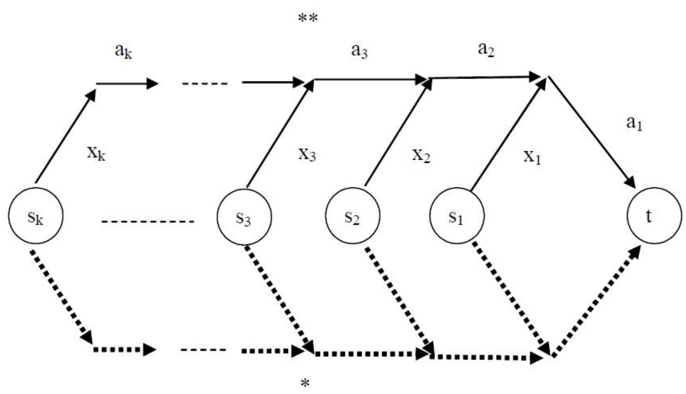

Figure 9: PNI implies SM.

perturbation of ** as following. The cost of path of an agent $j \neq i$ in both graphs is y_j. The cost of paths of agent i in "**" and "*" are y_i and $\tilde{y}_i$ where $\tilde{y}_i$ is in a neighborhood of y_i and $\tilde{y}_i > y_i$ and $|\tilde{y}_i - y_i| < \min\limits_{j,k \in K} |y_j - y_k|$. This restriction guarantees that the ranking will be preserved in the perturbed problem. Let the total cost of ** and * be $c - \epsilon$ and c respectively. First, we will show that this graph generates all feasible problems $(c; y)$. This happens if and only if the following system has a solution:

$$
\begin{aligned}
x_1 + a_1 &= y_1 \\
x_2 + a_2 + a_1 &= y_2 \\
x_3 + a_3 + a_2 + a_1 &= y_3 \\
&: \\
&: \\
x_k + a_k + a_{k-1} + ... + a_1 &= y_k \\
\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} a_i &= c \\
\forall i &\in K; \; x_i, a_i \geq 0
\end{aligned}
$$

We use Farka's Lemma to prove that this system indeed has a solution:

From Farka's Lemma we know that $Ax = b$; $x \geq 0$ has a solution if and only if $A^T z \geq 0$; $b^T z < 0$ doesn't have a solution.

Here, the $(k+1) \times (2k)$ matrix A, vector x and vector b are defined as follows:

$$
A = \begin{bmatrix} 1 & 0 & 0 & .... & 1 & 0 & 0 & .... \\ 0 & 1 & 0 & .... & 1 & 1 & 0 & .... \\ .... & .... & .... & .... & .... & .... & .... & .... \\ 1 & 1 & 1 & .... & 1 & 1 & 1 & .... \end{bmatrix}
$$

$x = \begin{bmatrix} x_1 & x_2 & .... & x_k & a_1 & a_2 & .... & a_k \end{bmatrix}^T$
$b = \begin{bmatrix} y_1 & y_2 & .... & y_k & c \end{bmatrix}^T$

$A^T z \geq 0$; $b^T z < 0$ gives the following $(2k+1)$ inequalities;

$$z_1 + z_2 + ... + z_{k+1} \geq 0 \tag{1}$$

$$z_2 + z_3 + ... + z_{k+1} \geq 0 \tag{2}$$

$$: \tag{:}$$

$$z_k + z_{k+1} \geq 0 \tag{k}$$

$$z_1 + z_{k+1} \geq 0 \tag{k+1}$$

$$z_2 + z_{k+1} \geq 0 \tag{k+2}$$

$$: \tag{:}$$

$$z_k + z_{k+1} \geq 0 \tag{2k}$$

$$y_1 z_1 + y_2 z_2 + ... + y_k z_k + c z_{k+1} < 0 \tag{2k+1}$$

Now, do the following operation on the first k inequalities: $y_1 \times (1) + (y_2 - y_1) \times (2) + ... + (y_k - y_{k-1}) \times (k)$, to get,

$$y_1 z_1 + y_2 z_2 + ... + y_k z_k + y_k z_{k+1} \geq 0 \tag{2k+2}$$

Now, for the inequalities $(2k+1)$ and $(2k+2)$ to be compatible, it must be the case that $z_{k+1} < 0$. Let this be the case and let $(2k+2)$ and $(2k+1)$ hold. Then, $(2k+1)$ implies:

$$y_1 z_1 + y_2 z_2 + ... + y_k z_k + (\sum_{i \in K} y_i) z_{k+1} < 0 \tag{2k+3}$$

This is true because feasibility requires $\sum_{i \in K} y_i \geq c$. Now, if we do the following operation on inequalities $(k+1)$ through $(2k)$: $y_1 \times (k+1) + y_2 \times (k+2) + ... + y_n \times (2k)$, then we get,

$$y_1 z_1 + y_2 z_2 + ... + y_k z_k + (\sum_{i \in K} y_i) z_{k+1} \geq 0 \tag{2k+4}$$

which contradicts $(2k+3)$ to give us the desired result.

We now prove the strong monotonicity of ξ. Clearly, the efficiency of ξ implies that ** is a NE but since * is a perturbation of **, we will have * as a NE for a perturbation small enough. The fact that ξ PNI the efficient graph implies the following inequality

$$\xi(c-\epsilon; y_i, y_{-i}) \leq \xi(c; \tilde{y}_i, y_{-i})$$

Using continuity we get,

$$\xi(c; y_i, y_{-i}) \leq \xi(c; \tilde{y}_i, y_{-i})$$

Now consider a perturbation where everything is exactly the same except ** costs $c+\epsilon$. Using the same argument of Pareto Nash implementability and continuity we get that

$$\xi(c; y_i, y_{-i}) \geq \xi(c; \tilde{y}_i, y_{-i})$$

Thus we conclude that $\xi(c; y_i, y_{-i}) = \xi(c; \tilde{y}_i, y_{-i})$ for $\tilde{y}_i$ in an open neighborhood of y_i. But repeatedly using the open neighborhood argument, show that this is true for any arbitrary y_i and $\tilde{y}_i$ as long as $(c; y_i, y_{-i})$ and $(c; \tilde{y}_i, y_{-i})$ are both feasible.

■

7.3.3 3. $\implies$ 4.

Consider a continuous ξ that implements the efficient graph in strong NE. Consider two arbitrary feasible problems $(c; y)$ and $(c; \tilde{y})$. We will prove that $\xi(c; y) = \xi(c; \tilde{y}) = f(c)$. The monotonicity of f comes from Lemma 3. Let $a = \frac{1}{k}\sum_{i\in K} y_i$ and $\tilde{a} = \frac{1}{k}\sum_{i\in K} \tilde{y}_i$. Assume without loss of generality that $y_1 \leq y_2 \leq y_3 \leq ...... \leq y_k$ and $\tilde{y}_1 \leq \tilde{y}_2 \leq \tilde{y}_3 \leq ...... \leq \tilde{y}_k$.

Step 1: $\xi(c; y) = \xi(c; a, a, ..., a)$ and $\xi(c; \tilde{y}) = \xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$

Proof:

Consider the following problems: $P_0 = (c; y)$, $P_1 = (c; a, y_2, y_3, ...y_k)$, $P_2 = (c; a, a, y_3, y_4, ..., y_k)$, ...,$P_k = (c; a, a, ..., a)$. Notice first that the feasibility of P_0 implies the feasibility of $P_1, P_2, ..., P_k$. This is true because the maximum of the demand profile doesn't go above y_k in all these problems and the sum of the individual demands is always at least $k * a = \sum_{i\in K} y_i$. Similarly, if we define the counterpart problems $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, ..., \tilde{P}_k$ where $\tilde{P}_i = (c; \tilde{a}, \tilde{a}, ..., \tilde{a}, \tilde{y}_{i+1}, \tilde{y}_{i+2}, ..., \tilde{y}_{k-1}, \tilde{y}_k)$, then again all of them will be feasible.

Now, due to the Separability Lemma, we must have $\xi_1(P_0) = \xi_1(P_1)$. Also, strong Nash implementability implies that $\xi_{-1}(P_0) = \xi_{-1}(P_1)$. To see this, suppose that it is not the case and for some agent $j \neq 1$, we have $\xi_j(P_0) \neq \xi_j(P_1)$. Assume without loss of generality that $\xi_j(P_0) < \xi_j(P_1)$. This means $\exists \grave{j} \in K\backslash\{1, j\}$ s.t., $\xi_{\grave{j}}(P_0) > \xi_{\grave{j}}(P_1)$, because of budget balancedness. Consider a network where all the agents $2, 3, ..., k$ have just one strategy which costs $y_2, y_3, ..., y_k$ and agent 1 has two strategies, where one of them costs y_1 and the other costs a. In both the cases, the total cost of the network is c. Thus one of the configurations generates the problem P_0 and the other P_1. Now both the configurations of the network are efficient and therefore at least one of them must be a strong NE under ξ. But clearly none of them is a strong NE. From P_1 the group $\{1, j\}$ has a profitable deviation and from

P_0 the group $\{1, \grave{j}\}$. Thus, we have $\xi(P_0) = \xi(P_1)$. Using the same argument we have $\xi(P_i) = \xi(P_{i+1})$ and $\xi(\tilde{P}_i) = \xi(\tilde{P}_{i+1})$ for all $0 \leq i \leq k-1$. Thus, we have $\xi(P_0) = \xi(P_k)$ and $\xi(\tilde{P}_0) = \xi(\tilde{P}_k)$ as desired.

Step 2: $\xi(c; a, a, ..., a) = \xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$

Proof:

Notice first that the feasibility of $(c; a, a, ..., a)$ & $\xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$ implies that any problem $(c; \hat{a})$ where some of the $\hat{a}_i = a$ and other $\hat{a}_i = \tilde{a}$ is also feasible. Now, the Separability Lemma implies $\xi_1(c; a, \tilde{a}, ..., \tilde{a}) = \xi_1(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$. And again, the strong Nash implementability implies $\xi_{-1}(c; a, \tilde{a}, ..., \tilde{a}) = \xi_{-1}(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$. The proof of this statement is analogous to the one in step 1. Thus we have $\xi(c; a, \tilde{a}, ..., \tilde{a}) = \xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a})$. Similarly, we get

$$\xi(c; \tilde{a}, \tilde{a}, ..., \tilde{a}) = \xi(c; a, \tilde{a}, ..., \tilde{a}) = \xi(c; a, a, \tilde{a}, ..., \tilde{a}) = ... = \xi(c; a, a, ..., a).$$

■

The results "4. $\Longrightarrow$ 1.," "4. $\Longrightarrow$ 2" and "4. $\Longrightarrow$ 3" are straightforward and the proof is omitted.

7.4 Proof of Theorem 2

Proof. The "if" part is clear. For, "only if" consider an arbitrary feasible problem $(c; y)$. Assume without loss of generality that $y_1 \geq y_2 \geq y_3 \geq .... \geq y_k$. Let $a = \frac{1}{k} \sum_{i=1}^{k} y_i$. Consider a problem $(c; a, a, ..., a)$ and suppose that ξ is continuous, efficient and satisfies ETE. Notice that the feasibility of $(c; y)$ implies the feasibility of $(c; a, a, ..., a)$ and any other problem $(c; \hat{y})$ where $\hat{y}_i = y_i$ for all $i \in \{1, 2, ..., l\}$ and $\hat{y}_i = a$ for all $i \in \{l, l+1, ..., k-1, k\}$. Now, the ETE property of ξ implies

$$\xi(c; a, a, ..., a) = (c/k, c/k, ..., c/k) \tag{4}$$

Using the Separability Lemma and applying ETE again we get,

$$\xi(c; y_1, a, ..., a) = (c/k, c/k, ..., c/k) \tag{5}$$

Now again applying the Separability Lemma and ETE we have,

$$\xi(c; y_1, y_2, a, a, ..., a) = (x_1, c/k, x, x, ..., x) \tag{6}$$

But if we change the ordering of 1 & 2 while arriving at the above profile then we should have,

$$\xi(c; y_1, y_2, a, a, ..., a) = (c/k, x_2, x, x, ..., x) \tag{7}$$

But, since the ordering is immaterial, we must have that $x_1, x_2, x = c/k$. And thus we have,

$$\xi(c; y_1, y_2, a, a, ..., a) = (c/k, c/k, ..., c/k) \tag{8}$$

Repeating the same argument, we conclude that $\xi(c; y) = (c/k, c/k, ..., c/k)$
■

8 Proof of Lemma 2

We show the existence of equilibrium for PR.

Proof. We prove a stronger property, which is that the best response (br) dynamics (one agent at a time) of any arbitrary fixed ordering of agents converges to a NE, no matter where we start the br dynamics from. Suppose, on the contrary, that for some fixed ordering of agents the br dynamics from some point "s" does not converge. This means that there is a cycle of a finite length $l - s(1) \to s(2) \to s(3) \to .... \to s(l) \to s(1)$. Say, without loss of generality, that this cycle includes deviations by the set of agents $M = \{1, 2, ..., m\} \subseteq K$. The strategy of agents in K/M is fixed at s^{-M}. Notice that l is at least as big as $2m$. This is so because after the l best responses, we arrive at the original strategy profile i.e., $s(1)$. Every agent in M is a part of the cycle, which in turn means that they change their strategy at least once. Therefore, it must be the case that every agent in M takes its turn at least twice so that they reach the original profile, i.e., $s(1)$. We assume that agent $i \in M$ takes its turn in the br dynamics $n_i > 1$ number of times so that $\sum_{i \in M} n_i = l$. Let the strategies played by the agent i in the cycle be $s^{i;1}, s^{i;2}, ..., s^{i;n_i}, s^{i;1}$ and so on. We call the agent who takes his turn of br in the movement from s_t to s_{t+1} agent a_t. Therefore, $s(1) = (s^{1;1}, s^{2;1}, ..., s^{m;1}, s^{-M})$, $s(2) = (s^{a_1;2}, s_{-a_1}(1))$, $s(3) = (s^{a_2;2}, s_{-a_2}(2))$, ...,$s(l-1) = (s^{a_{l-1};n_{a_{l-1}}}, s_{-a_{l-1}}(l-2))$, $s(l) = (s^{a_l;n_{a_l}}, s_{-a_l}(l-1))$. Here, we use the standard notation where $s_{-i}(t)$ represents the strategy profile of $K \backslash \{i\}$ fixed at that in $s(t)$. We abuse the notation and say that the cost of $s^{p;i}$ is equal to $s^{p;i}$. Here the cost of the network formed by the strategy profile $s(i) = C(G_{s(i)})$. Now, $\xi_j^{pr}(C(G_{s(i)}); s(i)) = s^{j;p} A_i$ where A_i is fixed for any particular $s(i)$ and $s^{j;p}$ represents the strategy of agent j in $s(i)$. The fixed A_i for an $s(i)$ is the ratio of $C(G_{s(i)})$ to the sum of the costs of individual paths in $s(i)$.

Now every step of the cycle corresponds to an inequality which we will present as follows:

Step 1: $s(1) \to s(2) \implies$

$$s^{a_1;2} \times A_2 < s^{a_1;1} \times A_1 \tag{1}$$

Step 2: $s(2) \to s(3) \implies$

$$s^{a_2;2} \times A_3 < s^{a_2;1} \times A_2 \tag{2}$$

Step 3: $s(3) \to s(4) \implies$

$$s^{a_3;t} \times A_4 < s^{a_3;t-1} \times A_3; t = \left\{ \begin{array}{c} 3 \; if \; a_3 = a_1 \\ 2 \text{ otherwise} \end{array} \right\} \tag{3}$$

|
|

Step p: $s(p) \to s(p+1) \implies$

$$s^{a_p;t} \times A_{p+1} < s^{a_p;t-1} \times A_p; t \in \{1, 2, ..., n_{a_p}\} \tag{p}$$

|
|
Step l: $s_l \to s_1 \implies$

$$s^{a_l;n_{a_l}} \times A_1 < s^{a_l;n_{a_l}-1} \times A_l \tag{l}$$

If we multiply the systems (2), (3), ..., (l) together[12], then everything else cancels out and we are left with $s^{a_1;2} \times A_2 > s^{a_1;1} \times A_1$, which contradicts the inequality (1). Therefore, we conclude that there cannot be any cycle regardless what ordering of agents and what initial point we follow for the best response dynamics. ■

9 Proof of Theorem 3

9.1 Any AEM meets WPNI

We start from an AEM φ determined by the functions $f^1, \ldots, f^k$.

Let $g^i = (f^i)^(-1)$ the inverse of f^i. Without loss of generality, we assume $g^1(s^1) \leq \cdots \leq g^k(s^k)$, where s_i is the cost of the stand-alone S^i of agent i.

Step 1. For any Nash equilibrium there is an index m and λ^* such that

i. $\varphi_i = s^i$ for $i = 1, \ldots, m$

ii. $\varphi_h = f^i(\lambda^*)$ for $h > m$.

iii. $g^m(s^m) < \lambda^* \leq g^{m+1}(s^{m+1})$

Proof. Consider an equilibrium $X = (X^1, \ldots, X^k)$ and let λ^* such that $\sum_i \min\{C(X^i), f^i(\lambda^*)\} = C(X^1, \ldots, X^k)$.

If agent j is such that $\varphi_j < f^j(\lambda^*)$, then $\varphi_j = s^j$. To see this, since $\varphi_j = \min\{C(X^j), f^i(\lambda^*)\} < \lambda^*$, then $\varphi_j = C(X^j)$. Since $C(X^j)) \geq s^j$, then agent j can deviate to his stand-alone S^j whenever $C(X^j)) > s^j$, and guarantee a cost-share not larger than s^j. Therefore at equilibrium $\varphi_j = s^j$.

Since $g^1(s^1) \leq \cdots \leq g^k(s^k)$, then there is m such that i and ii is satisfied. We now show that iii must be satisfied.

Assume that $\lambda^* > g^{m+1}(s^{m+1})$ then $f^{m+1}(\lambda^*) > s^{m+1}$, thus $\varphi_{m+1} > f^{m+1}(s^{m+1})$. Thus agent $m+1$ can profit by deviating from X by selecting S^{m+1}.

[12]Notice, we can do that since everything here is positive

Finally, since $C(X^i) = s^i$ for $i = 1, \dots, m$, and $\varphi_i = s^i$, then $s^i < f^i(\lambda^*)$ thus $g^i(s^i) < \lambda^*$.

■

Step 2. φ Pareto ranks the equilibriums.

Proof.

Consider any two equilibriums $X = (X^1, \dots, X^k)$ and $\tilde{X} = (\tilde{X}^1, \dots, \tilde{X}^k)$. Let (m, λ) and $(\tilde{m}, \tilde{\lambda})$ the values given by step 1 for equilibrium X and $\tilde{X}$ respectively.

If $m < \tilde{m}$, then $\lambda < \tilde{\lambda}$, since $g^m(s^m) < \lambda \leq g^{m+1}(s^{m+1}) \leq g^{\tilde{m}+1}(s^{\tilde{m}+1}) < \tilde{\lambda}$.

Hence, by step 1, agents $\{1, \dots, m\}$ are indifferent between both equilibriums and agents $\{m+1, \dots k\}$ prefer equilibrium X to $\tilde{X}$.

On the other hand, if $m = \tilde{m}$, then agents $\{1, \dots, m\}$ are indifferent between both equilibriums, and agents $\{m+1, \dots k\}$ rank the equilibrium depending on whether $\lambda < \tilde{\lambda}$ or vice versa.

■

9.2 WPNI implies AEG

We start the proof for two agents.

9.2.1 Proof for two agents

Step 1. The mechanisms satisfy truncation, that is $\varphi[c; y_1, y_2] = \varphi[c; \tilde{y}_1, \tilde{y}_2]$ for any $(\tilde{y}_1, \tilde{y}_2) \geq \varphi[c; y_1, y_2]$.

Proof.

Consider a feasible profile $(c; y_1, y_2)$ such that $y_1 + y_2 > c$. Let (p_1, p_2) such that $\varphi(c; y_1, y_2) = (p_1, p_2)$ and assume without loss of generality that $p_1 < y_1$. Consider $p_1 < \tilde{y}_1 < y_1$.

Construct networks depicted in figure 10 such that agent 1 has two strategies with costs y_1 and $\tilde{y}_1$, agent 2 also has two strategies with the same cost y_2, and the cost of the networks are $(c + \epsilon; y_1, y_2)$ and $(c; \tilde{y}_1, y_2)$.

Clearly the graphs that generate $(c + \epsilon; y_1, y_2)$ and $(c; \tilde{y}_1, y_2)$ are a Nash equilibrium for small ϵ.

By WPNI: $\varphi(c + \epsilon; y_1, y_2) \geq (c; \tilde{y}_1, y_2)$.

As ϵ tends to zero, and using continuity:

$$\varphi(c; y_1, y_2) \geq \varphi(c; \tilde{y}_1, y_2). \tag{9}$$

Similarly, consider the network in figure 11 such that agents 1 has two strategies with costs y_1 and $\tilde{y}_1$, agent 2 also has two strategies with the same cost y_2, and the cost of the networks are $(c; y_1, y_2)$ and $(c + \epsilon; \tilde{y}_1, y_2)$.

Clearly those two graphs are Nash equilibriums for small ϵ.

By WPNI: $\varphi(c + \epsilon; \tilde{y}_1, y_2) \geq \varphi(c; y_1, y_2)$.

As ϵ tends to zero, and using continuity:

$$\varphi(c; \tilde{y}_1, y_2) \geq \varphi(c; y_1, y_2). \tag{10}$$

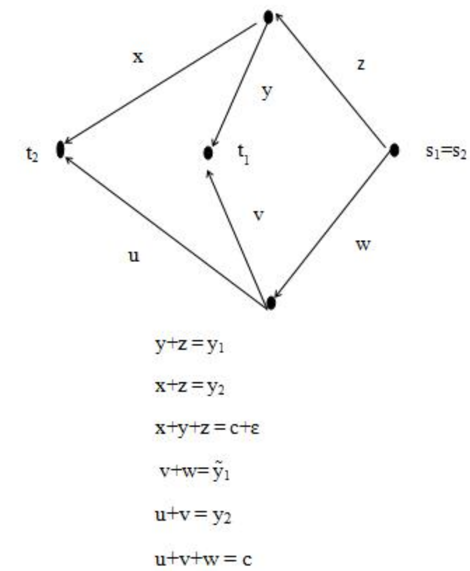

Figure 10: WPNI implies AEG (part 1).

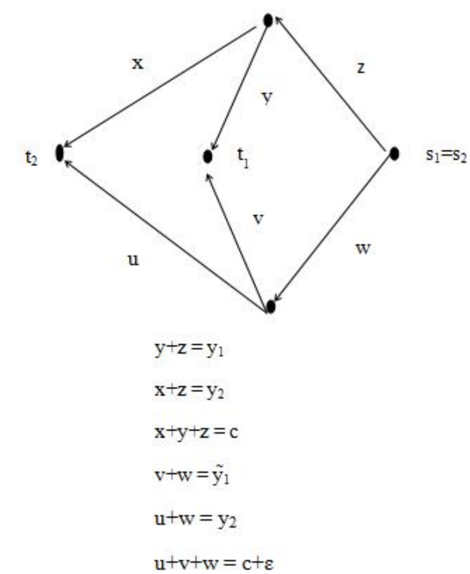

Figure 11: WPNI implies AEG (part 2).

by equations 9 and 10:

$$\varphi(c; \tilde{y}_1, y_2) = \varphi(c; y_1, y_2).$$

■

For any $c > 0$, let $g(c) = \varphi[c; c, c]$.

Step 2. $\varphi[c; y_1, y_2] = g(c)$ if $(y_1, y_2) \geq g(c)$; $= (y_1, c - y_1)$ if $y_1 < g_1(c)$; $= (c - y_2, y_2)$ if $y_2 < g_2(c)$.

Proof.

By step 1 and continuity, $\varphi[c; y_1, y_2] = g(c)$ if $(y_1, y_2) \geq g(c)$.

Consider $y = (y_1, c)$ such that $y_1 < g_1(c)$; and let $(p_1, p_2) = \varphi[c; y_1, c]$. Assume that $p_1 < y_1$.

By continuity, $\varphi_1(y_1 + \epsilon, c) \to p_1$ as ϵ tends to zero. Let $\tilde{\epsilon}$ by such that $\tilde{p}_1 = \varphi_1(y_1 + \tilde{\epsilon}, c) < y_1$.

Consider the demand $(\frac{\tilde{p}_1 + y_1}{2}, c)$, by truncation $\varphi(c; \frac{\tilde{p}_1 + y_1}{2}, \tilde{y}_2) = (\tilde{p}_1, c - \tilde{p}_1)$, for any $y_2 > c - p_1$.

Similarly, $\varphi(c; \frac{\tilde{p}_1 + y_1}{2}, \tilde{y}_2) = (p_1, c - p_1)$, which is a contradiction.

By truncation, $\varphi_1(y_1 + \tilde{\epsilon}, c) < y_1$.

■

Step 3. The mechanism should be weakly monotonic at the truncation point. That is $g(c) < g(\tilde{c})$ for $c < \tilde{c}$.

Proof.

Suppose that the mechanism is not weakly monotonic at the truncation point. Then, for any small ϵ we can find c and $c + \epsilon$ such that $g_2(c) > g_2(c + \epsilon)$ and $g_1(c) > g_1(c + \epsilon)$ (or vice versa).

Pick small ϵ and $b >> \max\{g(c), g(c + \epsilon)\}$ and $c > b_1 + b_2$.

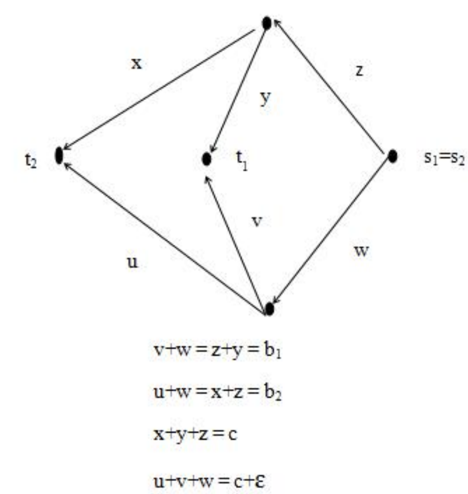

Figure 12: Monotonicity at truncation point.

Consider the network depicted in figure 12 such that every agent has two strategies, of equal cost b_1 and b_2, and generate problems $(c; b_1, b_2)$ and $(c + \epsilon; b_1, b_2)$. Clearly there are only two equilibria with costs c and $c+\epsilon$ but they are not Pareto ranked since $\varphi(c;b_1,b_2)=g(c)$ and $\varphi(c+\epsilon;b_1,b_2)=g(c+\epsilon)$. ■

Step 4. The mechanism can be represented by the above functions.

Proof. Consider the AEM represented by the functions $g_i(c)$ as above. It is easy to show that this mechanism generates the mechanism as above.

Indeed, consider $(c;y)$ feasible. Then If $(y_1,y_2)\geq g(c)$, then $\varphi(c;y)=g(c)$.

If $y_1<g_1(c)$, then by truncation $\varphi_1(c;y)=g_1(c)$ and $\varphi_2(c;y)=c-g_1(c)$.

If $y_2<g_2(c)$, then by truncation $\varphi_2(c;y)=g_2(c)$ and $\varphi_1(c;y)=c-g_2(c)$.

■

9.2.2 Extension to more than two agents

Proof. Consider any Parametric solution with k agents, $k>2$. We can replicate the above arguments for a network of any two agents $\{i,j\}$ by setting $y_l=0$ for $l\neq i,j$ (demanding independent demands with cost zero). Thus, by the previous case, $F_i(\lambda,y_i)=\min\{y_i,g_i(\lambda)\}$ for some non-decreasing function $g_i(\lambda)$.

■

10 Proof of Corollary 4

10.1 $POS(EG)=H(k)$

Consider the efficient profile $P=(P^1,\ldots,P^k)$ with cost c^*. Assume without loss of generality that $C(P^1)\geq C(P^2)\geq\cdots\geq C(P^k)$. Let S^i the stand-alone path of agent i with cost s^i.

Let $p=EG(c^*;C(P^1),C(P^2),\ldots,C(P^k))$. Clearly $p_1\geq p_2\geq\cdots\geq p_k$, and $p_i\leq\frac{c^*}{i}$ for $i=1,\ldots,k$.

Let λ^* and m be such that $p_k\leq p_{k-1}\leq\cdots\leq p_{m+1}<\lambda^*=p_m\cdots=p_1$.

Let $\tilde{K}=\{i|s^i<\lambda^*\}$. That is, $\tilde{K}$ is the set of agents with stand-alone less than λ^*. Consider the profile $Q=(P_{(\tilde{K})^c},S_{\tilde{K}})$, that is the strategy from each agents in $\tilde{K}$ is replaced by his stand-alone path.

Clearly, if $i>m$ then $i\in\tilde{K}$, since $s^i\leq p_i<\lambda^*$. Therefore Q contains at least all the agents who are paying their demand at P, but might include others.

Let $\tilde{k}=|\tilde{K}|$ the cardinality of $\tilde{K}$. First, notice that

$$C(S_{\tilde{K}})\leq(\tilde{k}-m)\lambda^*+s^{m+1}+\cdots+s^k\leq(\tilde{k}-m)\frac{c^*}{m}+\frac{c^*}{m+1}+\cdots+\frac{c^*}{k}.$$

Therefore $C(S_{(\tilde{K})})\leq\frac{c^*}{k-\tilde{k}+1}+\cdots+\frac{c^*}{k}$.

Hence $C(Q)\leq c^*+C(S_{(\tilde{K})})\leq c^*+\frac{c^*}{k-\tilde{k}+1}+\cdots+\frac{c^*}{k}$.

We repeat the above algorithm consecutively to the profile Q. That is, we find λ and move all the agent with stand-alone cost less than λ to their stand-alone path. Since there is at most k agents, this algorithm finishes in at most k steps. Let R the final profile of this algorithm.

From the above arguments, $C(R) \leq H(k)c^*$.

Let $\tilde{\lambda}$ the solution to the problem $EG(C(R); C(R_1), \ldots, C(R_k))$.

If $EG_i(C(R); C(R_1), \ldots, C(R_k)) = \tilde{\lambda}$ then $s_i \geq \tilde{\lambda}$.

On the other hand, if $EG_i(C(R); C(R_1), \ldots, C(R_k)) < \tilde{\lambda}$ then $R^i = S^i$.

Similarly to the existence of equilibrium for AEM, the best reply tattonment would converge to an equilibrium starting from the profile R, since λ and the cost would decrease at every step.

Indeed, if an agent is paying his stand alone, the only way to decrease his payment is by increasing his demand, and thus decreasing his cost. Therefore $\tilde{\lambda}$ should decrease. At his best reply, his stand-alone should be larger than the new λ.

On the contrary, if an agent is paying $\tilde{\lambda}$, then his best reply should decrease $\tilde{\lambda}$ because his stand-alone is larger than $\tilde{\lambda}$.

10.2 For any AEM ξ, $\xi \neq EG$, $PoS(\xi) > H(k)$

Proof. Consider the AEM mechanism ξ generated by the functions $f^1, \ldots, f^k$. Since $\xi \neq EG$, then there is i, j such that $f^i \neq f^j$.

Let λ^* be such that $f^i(\lambda^*) \neq f^j(\lambda^*)$, and c^* such that $c^* = f^1(\lambda^*) + \cdots + f^k(\lambda^*)$.

There is an agent l such that $f^l(\lambda^*) > \frac{c^*}{k}$, without loss of generality, assume such agent is agent k. That is, $f^k(\lambda^*) > \frac{c^*}{k}$. Let $\varphi_k^* = \varphi_k[c^*; c^*, \ldots, c^*] = f^k(\lambda^*) > \frac{c^*}{k}$.

Consider the problem $[c^* + f^k(\lambda^*); c^*, c^*, \ldots, f^k(\lambda^*)]$. Since

$$\varphi_k[c^* + f^k(\lambda^*); c^*, c^*, \ldots, f^k(\lambda^*)] \leq f^k(\lambda^*),$$

then there is an agent l, $l \neq k$, such that

$$\varphi_l[c^* + f^k(\lambda^*); c^*, c^*, \ldots, f^k(\lambda^*)] \geq \frac{c^*}{k-1}.$$

Without loss of generality, assume such agent is agent $k-1$. Let

$$\varphi_{k-1}^* = \varphi_{k-1}[c^* + f^k(\lambda^*); c^*, c^*, \ldots, f^k(\lambda^*)],$$

thus $\varphi_{k-1}^* \geq \frac{c^*}{k-1}$.

Consider the problem

$$[c^* + \varphi_{k-1}^* + f^k(\lambda^*); c^*, c^*, \ldots, c^*, \varphi_{k-1}^*, f^k(\lambda^*)].$$

Since

$$\varphi_k[c^* + \varphi_{k-1}^* + f^k(\lambda^*); c^*, c^*, \ldots, c^*, \varphi_{k-1}^*, f^k(\lambda^*)] \leq f^k(\lambda^*)$$

and

$$\varphi_{k-1}[c^* + \varphi_{k-1}^* + f^k(\lambda^*); c^*, c^*, \ldots, c^*, \varphi_{k-1}^*, f^k(\lambda^*)] \leq \varphi_{k-1}^*.$$

Then there is an agent l such that

$$\varphi_l[c^* + \varphi_{k-1}^* + f^k(\lambda^*); c^*, c^*, \ldots, c^*, \varphi_{k-1}^*, f^k(\lambda^*)] \geq \frac{c^*}{k-2}.$$

Without loss of generality, assume such agent is agent $k-2$. Let

$$\varphi^*_{k-2} = \varphi_{k-2}[c^* + \varphi^*_{k-1} + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi^*_{k-1}, f^k(\lambda^*)],$$

thus $\varphi^*_{k-2} \geq \frac{c^*}{k-2}$.

Continuing the same way, at step i, consider the problem

$$[c^* + \varphi^*_{i+1} + \dots + \varphi^*_{k-1} + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi^*_{i+1}, \dots, \varphi^*_{k-1}, f^k(\lambda^*)].$$

Since

$$\varphi_k[c^* + \varphi^*_{i+1} + \dots + \varphi^*_{k-1} + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi^*_{i+1}, \dots, \varphi^*_{k-1}, f^k(\lambda^*)] \leq f^k(\lambda^*)$$

and

$$\varphi_j[c^* + \varphi^*_{i+1} + \dots + \varphi^*_{k-1} + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi^*_{i+1}, \dots, \varphi^*_{k-1}, f^k(\lambda^*)] \leq \varphi^*_j,$$

for $j = k-1, \dots i+1$. Then there is an agent l such that

$$\varphi_l[c^* + \varphi^*_{i+1} + \dots + \varphi^*_{k-1} + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi^*_{i+1}, \dots, \varphi^*_{k-1}, f^k(\lambda^*)] \geq \frac{c^*}{i}.$$

Without loss of generality, assume such agent is agent i. Let

$$\varphi^*_i = \varphi_i[c^* + \varphi^*_{i+1} + \dots + \varphi^*_{k-1} + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi^*_{i+1}, \dots, \varphi^*_{k-1}, f^k(\lambda^*)].$$

Thus, $\varphi^*_i \geq \frac{c^*}{i}$.

Consider the network in figure 13.

Since $\varphi^*_i \geq \frac{c^*}{i}$ for $i = 1, \dots, k-1$ and $\varphi^*_k > \frac{c^*}{k}$, then the only equilibrium is where agent i demands the link (s_i, t) with cost φ^*_i. This equilibrium is inefficient and has a cost equal to $\sum_{i=1}^{k} \varphi^*_i > H(k)(c^* + \epsilon)$, for small ϵ.

■

10.3 Any IR mechanism has a PoS at least $H(k)$

Proof.

We show by an example that any individually rational cost-sharing rule must have a PoS of at least $H(k)$. Consider a situation as shown in figure 14. Here, every agent i has two strategies– either connect its demand nodes directly where the cost of the path is $1/i$ or connect through the path where link costs are 0 and $1+\epsilon$. Consider any arbitrary cost-sharing method ξ that satisfies individual rationality. We will show that if there exist an equilibrium, then this is where every agent is using its direct path to t. We prove this by contradiction.

Case 1. Assume all the agents use a free link to v and then the common link of cost $1+\epsilon$ to t. But then at least one of the agents must be paying more than $1/k$. We assume that this

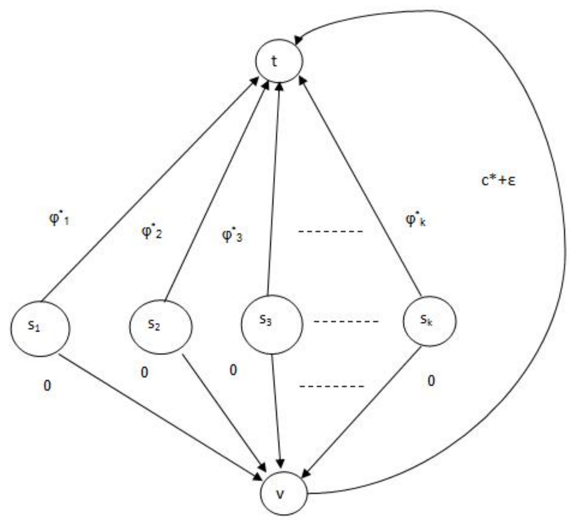

Figure 13: Optimality of the EG mechanism.

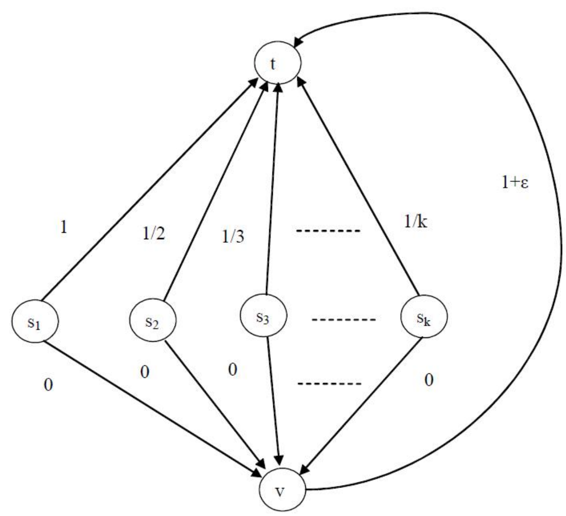

Figure 14: Incompatibility of EFF and IR

agent is the k-th agent in some configuration[13] of the graph. Then he will have a profitable deviation to go to the direct link of cost $1/k$ under any individually rational rule.

Case 2. Assume s agents are using their direct link and $k - s$ agents are sharing the common link to v. Then it follows from the individual rationality of the s agents that at least one of the remaining $k - s$ agents must be paying more than $1/(k - s)$. Notice that

[13]It is important to note that just one such configuration is enough, since PoS is a measure of the performance of the best NE in the worst case example.

in this case there exists an unused direct link, say, $s_j \to t$, of cost $1/s_j$ which is at most $1/(k-s)$. Now in some configuration of the graph, agent j will be the agent who is paying the above mentioned amount of more than $1/(k-s)$ and thus he would like to deviate.

We have just shown that no configuration different than the direct connection is a Nash equilibrium. If the equilibrium exists, then it must be the direct connection and has a cost equal to $H(k)$, whereas the efficient graph has a cost equal to $1+\epsilon$ (everyone uses a costless link to node v and then the common link to t). As ϵ goes to zero, the price of stability approaches to $H(k)$.

Finally, if there is no equilibrium then the price of stability equals to infinity.

■

10.4 Lower bound for PoS(PR)

Proof. Consider the network as shown in figure 15. We show that the unique equilibrium of the proportional method is of order k. Let the costs of links $s_i \to t$ be x_i. Straightforward computations show that the $k-th$ agent will deviate from the efficient graph of cost $1+\epsilon$ if $x_k \leq \frac{1-k+\sqrt{(k-1)^2+4k(k-1)}}{2k}$. As k grows, x_k converges to the golden number $\frac{\sqrt{5}-1}{2}$ in contrast to $1/k$ for the uniform method, which goes to zero. Also $x_{t-1} > x_t$ for all $t = 2, 3, .., k$ and $x_1 = 1$. Thus the lower bound on the PoS of the proportional method is $\sum_{i=1}^{k} x_i$, which is of order k. ■

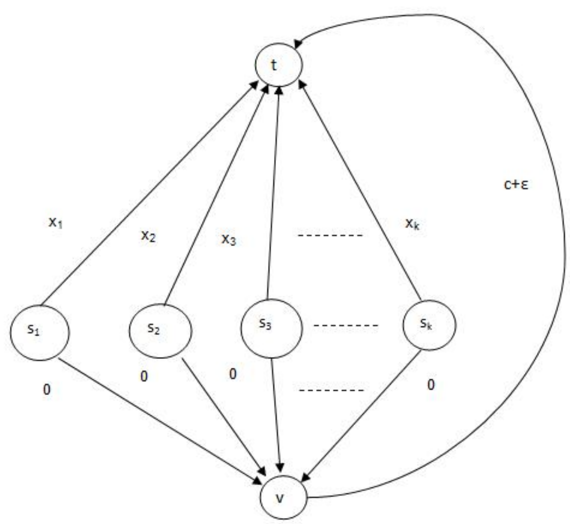

Figure 15: PoS(PR) is of order k.

11 Other proofs

11.1 Proof of Proposition 2

Proof.

The proof is very similar to the existence of Nash equilibrium for AEG in Lemma 2 and details are omitted to avoid repetition.

By step 1 in section 9.1, there exist an index m and λ such that

i. $\varphi_i = s^i$ for $i = 1, \ldots, m$

ii. $\varphi_h = f^h(\lambda)$ for $h > m$.

iii. $g^m(s^m) < \lambda \leq g^{m+1}(s^{m+1})$

Assume there are deviations by a group of agent from the equilibrium with the minimal cost c^1. Then λ will decrease at every step. If we continue iterating until there is no deviations, then in a finite number of iterations we converge to a profile Y such that agents either pay a common value $\tilde{\lambda}$ or their stand-alone cost $f^i(s^i)$. Since no group of agents can deviate, then Y is a Nash equilibrium of cost smaller than c^1, which is a contradiction.

■

11.2 The game generated by EG does not admit an ordinal potential

Consider a network shown in figure 16 below. Here there are two agents- agent 1 and agent 2 with their demand nodes being $\{s_1, t\}$ and $\{s_2, t\}$ respectively. Both the agents have two strategies each. One of the strategies of the agent i ($i = 1, 2$) is to connect through the direct link, i.e $s_i \to t$ and her other strategy is to connect indirectly through the node v, i.e., $s_i \to v \to t$. We denote the two strategies of agent 1 as a & b and the two strategies of agent 2 as c & d where $a := s_1 \to v \to t$, $b := s_1 \to t$, $c := s_2 \to t$ and $d := s_2 \to v \to t$. Given the EG mechanism, the game induced by the network on the set of agents can be represented in normal form by the following matrix, where the agent 1 is the row player and agent 2 is the column player. The first numbers in each cell of the matrix correspond to the cost share (negative of payoff) of agent 1 and the second to that of the agent 2.

	c	d
a	$3, 1.5$	$1.5, 1.5$
b	$2, 1.5$	$2, 1.5$

Suppose, that this is an ordinal potential game. Then, there must exist an ordinal potential function $P : \{a, b\} \times \{c, d\} \to \mathbb{R}$ satisfying $P(a, c) > P(b, c) = P(b, d) > P(a, d) = P(a, c)$ which is impossible.

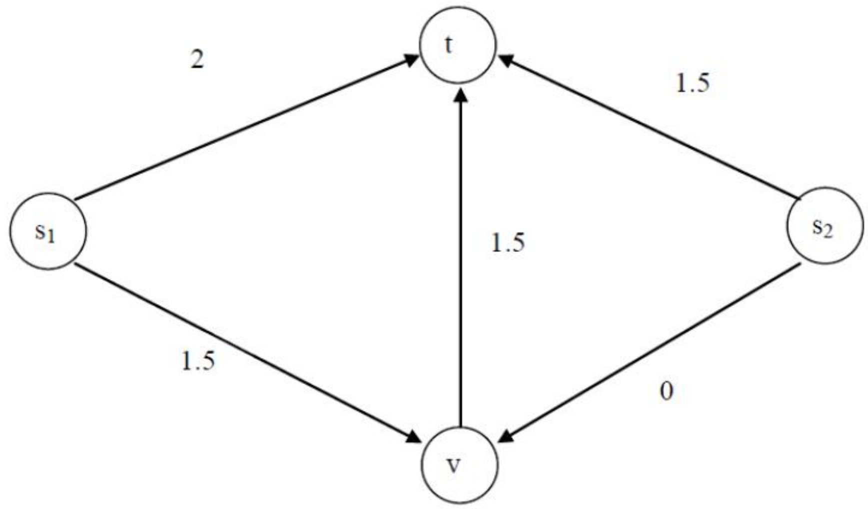

Figure 16: Network illustrating EG does not admit a potential.

References

[1] Andelman, N., Feldman, M., and Mansour, Y., 2009. Strong price of anarchy. Games and Economic Behavior 65, (2009) 289–317

[2] Anshelevich E., Dasgupta A., Kleinberg L., Tardos E., Wexler T., and Roughgarden T., 2004. The Price of Stability for Network Design with Fair Cost Allocation. In 45th Annual IEEE Symposium on Foundations of Computer Science (FOCS), pages 59-73, 2004.

[3] Aumann, R., M. Maschler, 1985. Game theoretic analysis of a bankruptcy problem from the Talmud. Journal of Economic Theory 36 195-213.

[4] Calcagno R., S. Lovo 2010. Preopening and Equilibrium Selection. Mimeo HEC Paris.

[5] Chambers, C. 2006. Asymmetric Rules for Claims Problems without Homogeneity. Games and Economic Behavior 54 (2006), 241-260.

[6] Chen, H. L., Roughgarden, T., Valiant, G., 2008. Designing Networks with good Equilibria. Proceedings of the 19th annual ACM-SIAM Symposium on Discrete Algorithms.

[7] Dagan, N., O. Volij, 1997. Bilateral comparisons and consistent fair division rules in the context of bankruptcy problems. International Journal of Game Theory 26, 11-25.

[8] Cooper, R., D. V. DeJong, R. Forsythe, T.W. Ross, 1992. Communication in Coordination Games. Quarterly Journal of Economics 107 (2) 739-771.

[9] Epstein, A., Feldman, M. and Mansour, Y., 2007. Strong Equilibrium in cost sharing connection games. ACM conference on Electronic Commerce, San Diego.

[10] Epstein, A., Feldman, M. and Mansour, Y., 2009. Strong Equilibrium in cost sharing connection games. Games and Economic Behavior 67, (2009) 51-68.

[11] Fiat, A., Kaplan, H., Levy, M., Olonetsky, S., and Shabo, R. 2006. On the price of stability for designing undirected networks with fair cost allocations. ICALP06.

[12] Green, J., and Laffont, J. 1979. Incentives in public decision-making. North-Holland Publishing Company.

[13] Hougaard, J. L. and Tvede, M. 2010. Truth-Telling and Nash Equilibria in Minimum Cost Spanning Tree Models. Mimeo Department of food and resource Economics, University of Copenhagen.

[14] http://www.foxnews.com/story/0,2933,475507,00.html

[15] http://www.msnbc.msn.com/id/28472161/

[16] http://archives.chicagotribune.com/2009/jan/05/nation/chi-oregon-gasjan05

[17] http://watchingthewatchers.org/indepth/24150/gas-vs-mileage-tax

[18] http://www.kansascity.com/105/story/1299981.html

[19] http://www.miamiherald.com/news/florida/AP/story/1281003.html

[20] Juarez, R. 2008. Group strategyproof cost sharing: the role of indifferences. Under Review.

[21] Juarez, R. 2008. The worst absolute surplus loss in the problem of commons: Random Priority vs. Average Cost. Economic Theory 34.

[22] Kaminski, M. 2000. Hydraulic rationing. Mathematical Social Sciences 40, 131-155.

[23] Kaminski, M 2006. Parametric rationing methods. Games and Economic Behavior, 54, 115-133.

[24] Kim, Y. 1996. Equilibrium Selection in n-Person Coordination Games. Games and Economic Behavior 15, 203-227.

[25] E. Koutsoupias, C. H. Papadimitriou, 1999. Worst-case equilibria. STACS 99.

[26] Kumar, R. 2009. Inefficiency of Shapley Mechanism and some alternatives. Mimeo Rice University.

[27] Jian Li., 2009. An upper bound on the price of stability for undirected Shapley network design games. Information Processing Letters 2009.

[28] Marden, J. R., Wierman, A., 2008. Distributed Welfare Games. Forthcoming in Operations Research.

[29] Maskin, E., T Sjostrom. In K. J. Arrow, A. K. Sen, K. Suzumura (ed.). Handbook of Social Choice and Welfare, 237-282.

[30] Monderer, D., and L. Shapley 1996. Fictitious play property for games with identical interests. Journal of Economic Theory 68, 258-65.

[31] Monderer, D, and L. Shapley, 1996. Potential Games. Games and Economic Behavior, 14, 124-43.

[32] Moulin, H. Incremental Cost Sharing: characterization by coalitional strategy-proofness. Soc. Choice Welfare 16 (1999), 279-320.

[33] Moulin, H., 2000. Priority rules and other asymmetric rationing methods. Econometrica 68 (3) 643-684.

[34] Moulin, H., 2008. The price of anarchy of serial, average and incremental cost sharing. Economic Theory, 36, 379-405.

[35] Moulin, H., 2009. Pricing traffic in spanning network. Proceedings of the ACM conference on Electronic Commerce, Stanford.

[36] Moulin, H. and Shenker, S. 2001. Strategyproof Sharing of Submodular Costs: Budget Balance versus Efficiency. Economic Theory, 18, 3, 511-533.

[37] Moulin, H., 2002. Axiomatic cost and surplus sharing. In K. J. Arrow, A. K. Sen, K. Suzumura (ed.). Handbook of Social Choice and Welfare, 289-357.

[38] O'Neill, B., 1982. A problem of rights arbitration from the Talmud," Mathematical Social Sciences 2, 345-371.

[39] Sharkey, W. W., Network models in economics, in: Ball, M. O., Magnanti, T. L., Nonma, C. L., Nemhauser, G. L. (Eds.), Handbooks in Operation Research and Management Science, 1995, 713-765.

[40] Saijo, T., Sjöström, T., Yamato, T., (2007). Secure implementation. Theoretical Economics 2, 203–229.

[41] Sandholm, W., 2001. Potential games with continuous player sets. Journal of Economic Theory, 97, 81-108.

[42] Sprumont, Y. 1991. The division problem with single-peaked preferences: A characterization of the uniform allocation rule. Econometrica, 59, 2: 509-519.

[43] Thomson, W. 2003. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Mathematical Social Sciences 45 249–297.

[44] Thomson, W. 2007. On the existence of consistent rules to adjudicate conflicting claims: a constructive geometric approach. Review of Economic Design 11(3) 225-251.

[45] Young, P. 1987. On dividing an amount according to individual claims or liabilities. Mathematics of Operations Research, 12: 398-414.

[46] Young, P., 1988. Distributive Justice in Taxation. Journal of Economic Theory 48, 321-335.