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Group strategyproof cost sharing: the role of indifferences

by
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Group strategyproof cost sharing: the role of indifferences

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Summary. Every agent reports his willingness to pay for one unit of good. A mechanism allocates goods and cost shares to some agents. We characterize the group strategyproof (GSP) mechanisms under two alternative continuity conditions interpreted as tie-breaking rules. With the maximalist rule (MAX) an indifferent agent is always served. With the minimalist rule (MIN) an indifferent agent does not get a unit of good.

GSP and MAX characterize the cross-monotonic mechanisms. These mechanisms are appropriate whenever symmetry is required. On the other hand, GSP and MIN characterize the sequential mechanisms. These mechanisms are appropriate whenever there is scarcity of the good.

Our results are independent of an underlying cost function; they unify and strengthen earlier results for particular classes of cost functions.

Keywords: *Cost sharing, Mechanism design, Group strategyproof, Tie-breaking rule.*

JEL classification: C72, D44, D71, D82.

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1 Introduction

Units of a nontransferable, indivisible and homogeneous good (or service) are available at some non-negative cost. Agents are interested to consume at most one unit of that good and are characterized by their valuation for it (which we call their utility). We look for mechanisms that elicit these utilities from the agents, allocate some goods to some agents and charge some money only to the agents who are served.[1]

These mechanisms have been widely explored in the cost-sharing literature (see below). The canonical example is sharing the cost of providing some optional service to geographically dispersed agents (e.g. Internet), where the cost function is not necessarily symmetric. Another example is auctions where the seller has multiple copies of a good.

When agents have private information about their utility, incentive compatibility of the mechanism, here interpreted as strategyproofness (SP), is an issue. The mechanisms that satisfy SP are such every agent is offered to buy a unit of good at a price that depends exclusively on the reports of the other agents.

A familiar strengthening of SP is group strategyproofness (GSP). This property rules out coordinated misreports of any group of agents. GSP is particularly interesting in settings where the designer of the mechanism has little information about the types of agents participating in the economy, for instance when the designer is dealing with agents in a large network like the internet. In these settings, it is usually the case that agents have the ability to coordinate misreports, and hence increase their net-utility. GSP is a robust property that rules out coordinated misreports under any possible information context. In particular, it works whether the information on individual characteristics is private or not.

For a SP mechanism, whether or not the agents who are offered a price equal to their valuation are served is of no consequence. Not so for GSP mechanisms. GSP is clearly violated if such an agent can be "bossy," i.e. affect the welfare of another agent without altering his own.[2] For instance, consider the mechanism that offers to the agents in $\{1, 2\}$, following the order $1 \succ 2$, the first unit at price p and the second unit at price p', $p' > p$. Assume the first agent's utility for a unit of good equals exactly p and the second agent's utility is strictly larger than p, then GSP requires agent 1 not to be served. Otherwise, agent 1 can help agent 2 by reporting a utility below p. Whereby agent 2 is offered the cheaper price p.

This paper characterizes the GSP mechanisms under two continuity conditions, interpreted as tie-breaking rules. With the maximalist tie-breaking rule (MAX), an agent who is indifferent between getting or not getting a unit of good will always get a unit of good. With the minimalist rule (MIN), the indifferent agents never get a unit of good.

The mechanisms that satisfy GSP and MAX are the cross-monotonic mechanisms (Theorem 1), where unlike in the above example the price offered to an agent weakly decreases as

[1] We also restrict the attention to individually rational mechanisms, that is such that no agent pays more that his utility.

[2] In some contexts, GSP is equivalent to the combination of SP and non-bossiness: Papai[2000, 2001], Ehlers et al.[2003], Svensson et al.[2002]. In our context, a similar equivalence holds by imposing two alternative non-bossy conditions, see Mutuswami[2005].

more agents are served. Specifically, for any subset of agents S consider a vector of nonnegative payments $x^S \in [0,\infty]^N$ that are zero for all agents not in S. A collection of payments is cross-monotonic if the payments are weakly inclusion decreasing. Given a cross-monotonic collection of payments, we construct the mechanism as follows. For a report of utilities allocate S^* at cost x^{S^*}, where S^* is the largest coalition of agents such that everyone in S^* is willing to pay x^{S^*} to get service –this coalition exists by cross-monotonicity of the payments.

The mechanisms that satisfy GSP and MIN are the sequential mechanisms (Theorem 2). Loosely speaking, consider any binary tree of size n such that to every node is attached exactly one agent and any path from the root to a terminal node goes through all agents exactly once. At every decision node we also attach a nonnegative price. Given this tree, we construct the mechanism as follows. First we offer service to the root agent at the price attached to his node. We proceed on the right branch from the root if service is purchased and on the left branch if it is not. The key restriction on prices is that for any two nodes to which the same agent is attached, the price on the winning node is not smaller than that on the losing node.[3]

Surprisingly, the (welfarewise) intersection of sequential and cross-monotonic mechanisms is almost empty. It contains only the fixed cost mechanisms (Corollary 1), offering to each agent a price completely independent of the reports.

An important property of cross-monotonic mechanisms is to allow equal treatment of equals, which no other GSP mechanism does (Proposition 2). On the negative side, when there are only k units of good available, $k < n$, cross-monotonic mechanisms must exclude $n-k$ agents from the mechanism, that is they will never be served at any profile (see section 6.3). By contrast, not all sequential mechanisms exclude agents ex-ante. In fact, only the priority mechanisms, where agents are offered sequentially a unit of good at a fixed price until someone accepts the offer, meet GSP and allocate at most one unit of good at any profile (Proposition 3).

We do not make an actual cost function part of the definition of a mechanism. That is, we place no constraint on the total cost shares collected from the agents who are served. Thus our characterization results of GSP mechanisms are entirely orthogonal to budget balance and other feasibility requirements (such as bounds on the budget surplus or deficit). Naturally, one of the first questions we ask about the class of mechanisms identified in theorems 1 and 2 is when can they be chosen so as to cover exactly a given cost function. In examples 7 and **??** we answer these questions under a weak symmetry assumption. In this way, we recover most mechanisms identified in the earlier literature.

2 Related literature

There is some interesting literature in the design of GSP mechanisms for assignment problems of heterogeneous goods when money is not available (Ehlers[2002], Ehlers et al.[2003], Papai [2000, 2001] and Svensson et al.[2002]). Unfortunately, this literature usually charac-

[3]See definition 9 for precise conditions.

terizes mechanisms with poor equity properties (e.g. dictatorial mechanisms). By contrast, the class of GSP mechanism when money is available is very rich (see below).

The design of GSP cost sharing mechanisms for heterogeneous goods was first discussed by Moulin[1999] and Moulin and Shenker[2001]. When the cost function is submodular (concave), cross-monotonic mechanisms are characterized by GSP, budget balance, voluntary participation, nonnegative transfers and strong consumer sovereignty.[4] Roughgarden et al.[2006a, 2006b], Pa'l et al.[2003] and Immorlica et al.[2005] consider cross-monotonic mechanisms when the cost function is not submodular. Roughgarden et al.[2006] uses submodular cross-monotonic mechanisms to approximate budget balance when the actual cost function is not submodular. Immorlica et al.[2005] shows that new cross-monotonic mechanisms emerge when consumer sovereignty is relaxed.

The sequential mechanisms of our Theorem 2 are discussed by Moulin[1999] who imposes budget balance for a supermodular (convex) cost function. Theorem 1 there asserts wrongly that all GSP mechanisms meeting budget balance, voluntary participation, nonnegative transfers and strong consumer sovereignty charge successively marginal cost following an independent ordering of the agents. We correct this erroneous statement in example 8.

Roughgarden et al.[2007] uncovers a very clever class of weakly GSP mechanisms that are neither cross-monotonic nor sequential (see also Devanur et al.[2005]). This class contains sequential and cross-monotonic mechanisms, as well as hybrid mechanisms. They apply these mechanisms to the vertex cover and Steiner tree cost sharing problems to improve the efficiency of algorithms derived from cross-monotonic mechanisms. A closely related paper is the companion paper Juarez[2007b] developing a model where indifferences are ruled out. For instance, agents report an irrational number and payments are rational. It turns out that the class of GSP mechanisms becomes very large. In particular, it contains mechanisms very different to cross-monotonic and sequential mechanisms (and also those discussed by Roughgarden et al.[2007]). Juarez[2007b] provides three equivalent characterizations of the GSP mechanism in this economy, two of which are generalizations of the cross-monotonic and sequential mechanisms discussed in this paper.

When a cost function is specified, an important question is to evaluate the trade-offs between efficiency and budget balance. Moulin and Shenker[2001] discuss this issue for budget balanced cross-monotonic mechanisms when the underlying cost function is submodular. In particular, they find that the cross-monotonic Shapley value mechanism, where the payment of a coalition equals its stand alone cost, minimizes the worst absolute surplus loss.[5] Juarez[2007a] analyzes similar trade-offs for supermodular cost functions. Contrary to the submodular case, one can construct optimal sequential mechanisms that cuts the efficiency loss by half with respect to the optimal budget balanced mechanism.

Finally a result by Goldberg et al.[2004] on fixed cost mechanisms is closely related to our Corollary 1. It characterizes these mechanisms under a strengthening of GSP, where agents can coalitionally manipulate by misreporting, transferring goods and money between

[4] Strong consumer sovereignty says that every agent has reports such that he gets (or does not get) a unit of good irrespective of other people reports.

[5] See also Juarez[2006] for a comparison of average cost and random priority using this measure. Moulin[2007] uses a similar measure to compare the serial, incremental and average cost methods.

them.

3 The model

For a vector $x \in \mathbb{R}^M$, we denote by x_S the projection of x over $S \subset M$. Let 1_M be the unitarian vector in $\mathbb{R}^M$, that is $1_M = (1, 1, \ldots, 1)$.

There is a finite number of agents $N = \{1, 2, \ldots, n\}$. Every agent has a utility (willingness to pay) for getting one unit of good. Let $u \in \mathbb{R}^N_+$ be the vector of these utilities. Therefore, if agent i gets a unit paying x_i, his net utility is $u_i - x_i$. If he does not get a unit his net utility is zero.

Definition 1 *A mechanism (G, φ) allocates to every vector of utilities u a coalition of agents who get goods $G(u) \subseteq N$ and the cost shares (payments) $\varphi(u) \in \mathbb{R}^N_+$ such that:*

i. if $i \notin G(u)$ then $\varphi_i(u) = 0$

ii. if $i \in G(u)$ then $\varphi_i(u) \leq u_i$.

The definition above includes familiar constraints previously discussed in the literature. For instance, we restrict the attention to non-negative mechanisms, requiring for all cost shares to be positive or zero. This is a common assumption when no transfers between agents are allowed and we do not want to subsidize any of them. The mechanisms will also meet individual rationality, which implies that all agents enter the mechanism voluntarily. That is, if an agent is served then he will never pay more than his utility (condition *ii*). On the other hand, because we study non-negative mechanisms, individual rationality implies the agents with zero utility should pay nothing (condition *i*).

The net utility of agent i in the mechanism (G, φ), denoted by NU_i, is $NU_i(u) = \delta_i(G(u))(u_i - \varphi_i(u))$.[6] That is, if $i \in G(u)$ then $NU_i(u) = u_i - \varphi_i(u)$; and if $i \notin G(u)$ then $NU_i(u) = 0$. Let $NU(u)$ be the vector of such net utilities. Notice two different mechanisms may be welfarewise equivalent, that is their net utilities at any profile be equal.

We want to characterize the mechanisms that are group strategyproof. That is, we want to rule out coordinated misreports by group of agents. If a group of agents misrepresent their preferences with at least one agent in the group strictly profiting, then another agent in the group will lose.

Definition 2 (Group strategyproofness) *A mechanism (G, φ) is group strategyproof (GSP) if for all $T \subset N$, and all utility profiles u and u' such that $u'_{N\setminus T} = u_{N\setminus T}$, if there is an agent $i \in T$ such that $(u_i - \varphi_i(u'))\delta_i(G(u')) > NU_i(u)$, then there exist another agent $j \in T$ such that $(u_j - \varphi_j(u'))\delta_j(G(u')) < NU_j(u)$.*

We define next our two systematic continuity conditions. Similar continuity conditions have been used in other models, for instance Deb and Razzolini[1999]. These are tractability conditions that allow us to get close-form mechanisms. Nevertheless, these conditions can be easily interpreted, see below.

[6] δ is the classic Dirac delta function, $\delta_i(T) = 1$ if $i \in T$, and 0 otherwise.

Definition 3 • ***Upper continuity (Maximalist tie-breaking rule (MAX)).*** *A mechanism (G, φ) satisfies MAX if for any i, $u_{-i} \in \mathbb{R}_+^{N \setminus i}$, and $u_i^1 \geq u_i^2 \geq \cdots \to u_i^*$ such that $i \in G(u_i^k, u_{-i})$ for all k, then $i \in G(u_i^*, u_{-i})$.*

- ***Lower continuity (Minimalist tie-breaking rule (MIN)).*** *A mechanism (G, φ) satisfies MIN if for any i, $u_{-i} \in \mathbb{R}_+^{N \setminus i}$, and $u_i^1 \leq u_i^2 \leq \cdots \to u_i^*$ such that $i \notin G(u_i^k, u_{-i})$ for all k, then $i \notin G(u_i^*, u_{-i})$.*

In the space of strategyproof mechanisms, upper and lower continuity can be interpreted as tie-breaking rules. Upper continuity (MAX) serves the agents who are indifferent between getting or not getting a unit of good, whereas lower continuity (MIN) does not serve the indifferent agents. To see this, consider a SP mechanism. Then, there exist arbitrary pricing functions $f_i : \mathbb{R}_+^{N \setminus i} \to [0, \infty]$ for $i = 1, \ldots, n$, such that at the utility profile u, agent i is offered a unit of good at price $f_i(u_{-i})$. That is, if $u_i > f_i(u_{-i})$ then i is served at price $f_i(u_{-i})$; if $u_i < f_i(u_{-i})$ then i is not served and pays nothing; and if $u_i = f_i(u_{-i})$ then i may get a unit of good at this price or may not get it. Under MAX, if $u_i = f_i(u_{-i})$ then the agent gets a unit of good at price $f_i(u_{-i})$. On the other hand, under MIN, if $u_i = f_i(u_{-i})$ then i does not get a unit of good and pays nothing.

4 Cross-monotonic mechanisms and MAX

Definition 4 *A cross-monotonic set of cost shares (payments) $\chi^N = \{x^S \in \mathbb{R}_+^N \mid S \subseteq N\}$ is such that:*

i. $x^S_{N \setminus S} = 0$ *for all* $S \subseteq N$, *and*

ii. *if* $S \subseteq T$ *then* $x_i^S \geq x_i^T$ *for all* $i \in S$

In a cross-monotonic set of cost shares, there is exactly one set of cost-shares for every coalition S. We interpret x^S as the payment when the agents in S, and only them, are served.

The key feature of a cross-monotonic set of cost shares is that they do not increase as the coalition increases. This implies that for every utility profile u the set of reachable coalitions, $F(u) = \{S \in 2^N \mid x^S \leq u\}$, has a maximum element with respect to the inclusion $\subseteq$. To see this, notice if $S, T \in F(u)$ then by cross-monotonicity $S \cup T \in F(u)$.

Definition 5 *A mechanism (G, φ) is cross-monotonic if there exists a cross-monotonic set of cost shares χ^N such that for all $u \in \mathbb{R}_+^N$: $G(u)$ is the maximum reachable coalition at u and $\varphi(u) = x^{G(u)}$.*

Theorem 1 *A mechanism satisfies GSP and MAX if and only if it is cross-monotonic.*

In an economy without indifferences, cross-monotonic mechanisms are also characterized by GSP and monotonicity in size, that is if $u \leq \tilde{u}$ then $G(u) \subseteq G(\tilde{u})$. See Juarez[2007b] for details.

Given a cross-monotonic set of cost shares χ^N, we can also implement the truthful outcome of the cross-monotonic mechanism by playing the following demand game proposed by Moulin[1999]. We offer agents in N units of good at price x^N. If all of they accept it, then everyone is served at prices x^N. If only agents in S accept, then we remove agents in $N \setminus S$ from the game and offer agents in S units of good at price x^S. Continue similarly until all of the agents in a coalition accepted or every agent in N was removed from the game.

Example 1 (Cross-monotonic mechanisms for $n = 1, 2$) *The one agent mechanisms can be described by a constant x, $x \in [0, \infty]$. The agent gets a unit and pays x if his utility is bigger than or equal to x. He does not get a unit and pays nothing otherwise.*

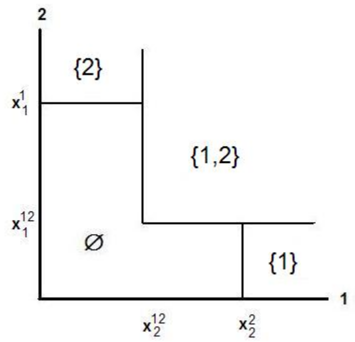

Figure 1: Generic form of a 2-agent cross-monotonic mechanism.

The two agent mechanisms should be generated by a cross-monotonic set of cost shares. Thus $0 \leq x_1^{\{1,2\}} \leq x_1^{\{1\}}$ and $0 \leq x_2^{\{1,2\}} \leq x_2^{\{2\}}$ (see figure 1).

By MAX, the level set of $\{1, 2\}$ is closed. The borders between the level sets of $\{1\}$ and $\emptyset$, and $\{2\}$ and $\emptyset$, should belong to the $\{1\}$ and $\{2\}$ respectively.

Example 2 *Immorlica et al.[2005] proposes an example where exactly one agent pays a positive amount when a coalition of agents is served. This example relaxes a key strong consumer sovereignty condition on Moulin[1999] result, therefore is not captured by Moulin's mechanisms. However, it is captured by our class of cross-monotonic mechanisms. For a submodular cost function $C : 2^N \rightarrow \mathbb{R}_+$, order the agent arbitrary, say $i_1 \succ i_2 \succ \cdots \succ i_n$. Offer the agents, following this order, a unit of good at the cost of himself and the agents after him. The mechanism ends when someone accepts the offer or when we have made an offer to every agent. That is, agent i_1 will be offer a unit at price $C(i_1, \ldots, i_n)$. If he accepts, the mechanism ends there. If he rejects, we offer agent i_2 a unit of good at price $C(i_2, \ldots, i_n)$, and so on. The cross-monotonic set of cost shares that implements this mechanism is $x_{i^*}^S = C(D_{i^*})$ and $x_j^S = 0$ for all $j \neq i^*$, where i^* is the maximal element in S and D_i^* is the set that contains i^* and all agents dominated by i^* with $\succ$.*

5 Sequential mechanisms and MIN

Definition 6 *A sequential tree is a binary tree of length n such that:*

i. at every node there is exactly one agent in N and a price in $[0, \infty]$,

ii. every path from the root to a terminal node contains all agents in N exactly once.

Definition 7 (Sequential mechanisms) *Given a sequential tree we construct a sequential mechanisms as follows:*

We offer the agent in the root of the tree a unit of good at the price of his node. If his utility is strictly bigger than the offered price, then we allocate him a unit at this price and go right on the tree. If his utility is smaller than or equal to the offered price then we do not allocate him a unit and go left on the tree. We continue similarly with the following agent until we reach the end of the tree.

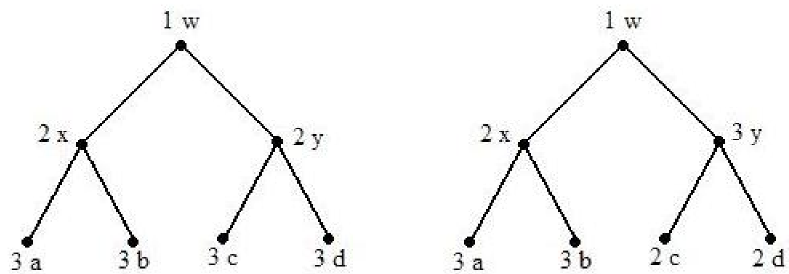

Figure 2: Sequential trees for three agents. (a) Agents follow order 1,2,3. (b) Agents 2 and 3 follow different orders depending on whether agent 1 go right or left.

Example 3 *In figure 2 we show the only two possible (up to renaming the agents) sequential trees for the agents in $N = \{1, 2, 3\}$. Every node contains a number and a letter. The number represent the agent in this node. The letter represent a prices in $[0, \infty]$.*

Consider the sequential tree of figure 2(a) and the mechanism (G, φ) that it implements. If the utility profile u is such that $u_1 > w, u_2 > y$ and $u_3 \leq d$ then the outcome is $G(u) = \{1, 2\}$ and $\varphi(u) = (w, y, 0)$.

On the other hand, if $\tilde{u}$ is such that $\tilde{u}_1 \leq w, \tilde{u}_2 > x$ and $\tilde{u}_3 \leq b$ then $G(\tilde{u}) = \{2\}$ and $\varphi(\tilde{u}) = (0, x, 0)$.

Sequential mechanisms are not always group strategyproof. For instance, consider the mechanism generated by the sequential tree of figure 2(a). If $y < x$, then when the true utility profile is such that $u_1 = w$ and $u_2 > y$, agent 1 can help agent 2 by reporting a utility bigger than w, whereby agent 2 is offered a unit at a cheaper price. However, these mechanisms are *weakly group strategyproof*, that is if a coalition of agents successfully misreports, then at least one agent in this coalition is indifferent. Definition 10 gives the exact conditions under which a sequential tree will generate a GSP mechanism.

Given a sequential tree, consider any path in the tree and a non terminal node ζ in this path. We say ζ is losing (winning) on this path if the edge in the path that follows ζ is a left (right) edge. For instance, the path $[1w, 2y, 3c]$ in figure 2(a) contains one winning node and one losing node. $1w$ is winning and $2y$ is losing.

One useful path is from the root of the tree to a node. We denote by $P_0(\zeta)$ this path starting at node ζ. For instance, in figure 2(a), $P_0(3c) = [1w, 2y, 3c]$, $P_0(3d) = [1w, 2y, 3d]$ and $P_0(2x) = [1w, 2x]$.

Notice the intersection of two paths from the root of the tree is also a path from the root of the tree. We use $\sqcap$ to denote it. For instance, in figure 2(a), $P_0(3c) \sqcap P_0(3d) = [1w, 2y]$. Notice this intersection may also lead to the degenerated path that contains only the root of the tree, for instance $P_0(2x) \sqcap P_0(2y) = [1w]$.

Definition 8 *Let ζ and ζ' two nodes in a sequential tree. We say the node ζ is on the left of ζ' if the terminal node of $P_0(\zeta) \sqcap P_0(\zeta')$ is losing on $P_0(\zeta)$ and winning on $P_0(\zeta')$.*

For instance, in figure 2(a), $P_0(3c) = [1w, 2y, 3c]$, $P_0(3d) = [1w, 2y, 3d]$. Since $2y$ is losing in $[1w, 2y, 3c]$ and winning in $[1w, 2y, 3d]$, then $3c$ is on the left of $3d$.

Given a node ζ and an agent $i \in P_0(\zeta)$, we denote by x_i^ζ the price of agent i in the path $P_0(\zeta)$.

Definition 9 (Realizability by indifferent agents) *Consider a sequential tree and two nodes ζ and ζ' in the tree. We say ζ' is realizable by indifferent agents from ζ if (a) ζ is on the left of ζ', (b) there is a utility profile that visits ζ, and (c) there is a group of indifferent agents S who can increase their utility profile without increasing their payment and visit the node ζ'.*

That is, there is a utility profile u that visits the node ζ, a group of agents S such that $u_S = x_S^\zeta \geq x_S^{\zeta'}$ and the profile $(\tilde{u}_S, u_{-S})$ visits ζ' for some $\tilde{u}_S > u_S$.

Two nodes are realizable by indifferent agents if there is a group of agents who can increase their utility without increasing their payment and visit the second node. For instance, consider the tree in figure 2(a). The node $3d$ is realizable by indifferent agents from $3b$ whenever x and y are finite. To see this, consider the utility profile $u = (w, u_2, u_3)$ such that $u_2 > \max\{x, y\}$. Clearly, u visits $3b$. If $\tilde{u}_1 > w$ then the profile $(\tilde{u}_1, u_2, u_3)$ visits $3d$.

On the other hand, if $y \leq x$, then the node $3c$ cannot be realizable be indifferent agents from $3b$. This is because at any utility profile that realizes $3b$, the utility u_2 of agent 2 is such that $u_2 > x \geq y$. Therefore agent 2 will always be served independent on the utility of agent 1.

Realizability by indifferent agents capture the nodes where indifferent agents can increase their utility without increasing their payment. We are especialy interested in realizability by indifferent of agents of nodes with a common agent.

Definition 10 *A sequential tree is feasible if for any nodes* ζ *and* ζ' *with a common agent* k *such that* ζ' *is realizable by indifferent agents from* ζ, $x_k^{\zeta} \leq x_k^{\zeta'}$.

We say a sequential mechanism is feasible if it is implemented by a feasible sequential tree.

A sufficient condition for feasibility is that for any two nodes that contain the same agent, the price of the node in the left is not smaller than the price of the node in the right. This condition is necessary when there are at most three agents (see examples 4 and 5). Example 6 shows this is not necessary when there are more than three agents. We now characterize the collection of sequential trees that are feasible.

Proposition 1 *A sequential tree is feasible if and only if for any two nodes* ζ *and* ζ' *with a common agent* k *such that* ζ *is on the left of* ζ' : *If* $x_k^{\zeta} > x_k^{\zeta'}$, *then there exist nodes* $\tilde{\zeta} \in P_0(\zeta)$ *and* $\bar{\zeta} \in P_0(\zeta')$ *with a common agent* i *and:*

(a) $\tilde{\zeta}$ *is losing in* $P_0(\zeta)$, $\bar{\zeta}$ *is winning in* $P_0(\zeta')$ *and* $x_i^{\zeta} < x_i^{\zeta'}$, *or*

(b) $\tilde{\zeta}$ *is winning in* $P_0(\zeta)$ *and* $\bar{\zeta}$ *is losing in* $P_0(\zeta')$ *and* $x_i^{\zeta} \geq x_i^{\zeta'}$.

Feasibility is a neccesary condition of a sequential mechanism that is GSP. Indeed, consider two nodes ζ and ζ' as in definition 10 above and assume that $x_k^{\zeta} > x_k^{\zeta'}$. We see that the indifferent agents can help agent k by moving from node ζ to node ζ' at some utility profile. Since ζ' is realizable by indifferent agents from ζ, then there is a utility profile u that visits ζ and a group of indifferent agents S such that $u_S = x_S^{\zeta}$, $x_S^{\zeta} \geq x_S^{\zeta'}$, and for some $\tilde{u}_S > u_S$ $(\tilde{u}_S, u_{-S})$ visits ζ'. Assume without loss of generality that $u_k > x_k^{\zeta'}$. Then the indifferent agents in S can help k when the true utility profile is u by misrepresenting their preferences to $\tilde{u}_S$. Agent k will get a good at the cheaper price $x_k^{\zeta'}$ whereas the indifferent agents in S are offered units of good at the cheaper price $x_S^{\zeta'}$ because $x_S^{\zeta'} \leq x_S^{\zeta}$.

We now state the second main theorem of the paper. It characterizes the class of GSP and MIN mechanisms. This class only contains the feasible sequential mechanisms.

Theorem 2 *A mechanism is* GSP *and* MIN *if and only if it is a feasible sequential mechanism.*

Example 4 (Feasible sequential mechanisms for $n = 1, 2$**)** *The one agent mechanisms are easy to describe. Given* $x_1 \in [0, \infty]$, *agent* 1 *gets a unit of good at price* x_1 *if and only if* $u_1 > x_1$.

A two agents mechanism such that 2 has priority over 1, is shown in figure 3. Agent 2 gets a unit of good at price x^2 *if and only if* $u_2 > x^2$. *If 2 gets a unit of good, then agent 1 gets a unit of good at price* d^1 *if* $u_1 > d^1$. *On the other hand, if agent 2 did not get a unit of good, then agent 1 gets a unit of good at price* d^2 *if* $u_1 > d^2$. *By feasibility of the tree* $d^2 \leq d^1$.

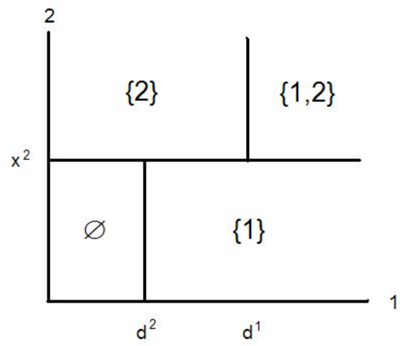

Figure 3: Generic form of a 2-agent feasible sequential mechanism.

The following example shows that any feasible sequential mechanism for three agents with finite prices can be represented by a sequential tree such that for any two nodes with a common agent, the price on the losing node is not larger than the price on the winning node. This property is true only for three or less agents. Example 6 provides a four agents example where this representation does not hold.

Example 5 (Feasible sequential mechanisms for $n = 3$) *Assume there are three agents. Figure 2 shows sequential trees for three agents. Every node contains an agent from $\{1, 2, 3\}$ and a nonnegative price.*

On figure 2(a), a feasible sequential tree (assuming finite prices) implies: $x \leq y$, $a \leq b \leq d$ and $a \leq c \leq d$. Also, if $x < y$ then $b \leq c$.

To see this, consider nodes $2x$ and $2y$. Since they are consecutive nodes, their paths to the root of the tree only differ in $2x$ and $2y$ respectively. Then, conditions (a) and (b) cannot be satisfied. Hence $x \leq y$.

Similarly, $a \leq b$ and $c \leq d$ are satisfied by comparing nodes $3a$ and $3b$, and $3c$ and $3d$ respectively.

On the other hand, by comparing nodes $3a$ and $3c$, conditions (a) and (b) are not satisfied because $2x$ and $2y$ are both losing. Hence $a \leq c$. Similarly $b \leq d$.

Now consider the nodes $3b$ and $3c$. If $x < y$, then condition (a) is not satisfied because $2y$ is not winning. Condition (b) is not satisfied because $x < y$. Therefore it cannot be that $b > c$. Hence $x < y$ and $a \leq b \leq c \leq d$.

Finally, assume $x = y$. From the argument given above, $a \leq b \leq d$ and $a \leq c \leq d$.

If $b \leq c$ then for every two nodes with the same agent, the price on the losing node is smaller than the price on the winning node.

On the other hand, if $b > c$ then because agents 1 and 2 have priority, we can exchange their order on the tree. This will look like figure 4. With this order, for every two nodes with the same agent, the price on the losing node is smaller than the price on the winning node.

Now consider the figure 2(b). Then feasibility of the tree (assuming finite prices) requires that $a \leq b \leq y$ and $x \leq c \leq d$. That is for every two nodes with the same agent, the price on the losing node is smaller than the price on the winning node.

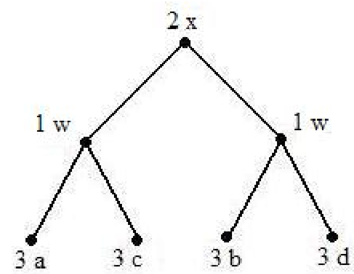

Figure 4: Three agents sequential tree such that the positions of agents 1 and 2 can be switched without affecting the final outcome.

To see this, by comparing nodes 3*a and* 3*b, and* 2*c and* 2*d, we get (similarly to example above) that* $a \leq b$ *and* $c \leq d$ *respectively.*

Now we compare nodes 3*b and* 3*y. Then there is no common agent in their path to the root, thus conditions (a) and (b) cannot be satisfied. Hence* $b \leq y$*. That is,* $a \leq b \leq y$*.*

Similarly, by comparing nodes 2*x and* 2*c,* $x \leq c$*. Hence* $x \leq c \leq d$*.*

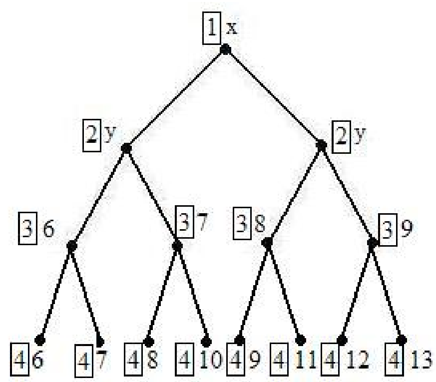

Figure 5: Four agents feasible sequential tree such that for every two nodes with the same agent, the price of the winning node may not be smaller than the price of the losing node.

Example 6 *Consider the mechanism generated by the sequential tree of figure 5 (agents are in the rectangles). For every two nodes with the same agent, the price on the losing node is not bigger than the price on the winning node, except for nodes (4 10) and (4 9). At these nodes, their paths to the root contain the common agent 2. This agent meets condition (b). Therefore this tree is feasible.*

However, the price on the losing node (4 10) is bigger than that on the winning node (4 9).

Since agents 1 and 2 have priority, we can also exchange their positions and leave agent agent 2 in the root. If this is the case, node (3 8) is on the left of (3 7).

6 Comparison between cross-monotonic and sequential mechanisms

6.1 The intersection of cross-monotonic and sequential mechanisms

There is a small class of mechanisms that are welfare equivalent to both a sequential and a cross-monotonic mechanism.[7]

Definition 11 *A mechanism (G, φ) is a fixed cost mechanism if there exist $x_1, \ldots, x_n \in [0, \infty]$, such that for every utility profile u:*

i. *if $u_i > x_i$ then $i \in G(u)$,*

ii. *if $u_i < x_i$ then $i \notin G(u)$,*

iii. *if $i \in G(u)$ then $\varphi_i(u) = x_i$*

iv. *if $i \notin G(u)$ then $\varphi_i(u) = 0$*

A fixed cost mechanism offers to agent i a unit of good at price x_i. Indifferences are broken arbitrarily. That is, for the utility profile u, agent i is guaranteed a unit at price x_i if $u_i > x_i$. Agent i does not get a unit if $u_i < x_i$. At $u_i = x_i$ he may or may not get a unit.

Corollary 1 *A mechanism is welfare equivalent to a cross-monotonic and a feasible sequential mechanism if and only if it is a fixed cost mechanism.*

This result shows that the behavior of indifferences have a big impact on the class of GSP mechanism. But one can argue that indifferences are rare event, so that a better model is one where the domain of utilities and the class of mechanisms preclude indifferences. On such domain, the class of GSP mechanisms will contain many more mechanisms than the sequential and cross-monotonic mechanisms. Juarez[2007b] analyzes such domain and characterizes the corresponding GSP mechanisms.

[7] The intersection of sequential and cross-monotonic mechanism only contain the trivial mechanism where no agent is served at any profile. This is because if agent i is served at some profile u paying $p_i \geq 0$, then at the profile (p_i, u_{-i}) agent i would be served under a cross-monotonic mechanism but not so under a sequential mechanism. Therefore no agent can be served at any profile.

6.2 Equal treatment of equal agents

Definition 12 *We say a mechanism satisfies equal treatment of equals (ETE) if for any* u *such that* $u_i = u_j$, $i \in G(u)$ *then* $j \in G(u)$ *and* $\varphi_i(u) = \varphi_j(u)$.

Proposition 2 *A mechanism meets* GSP *and* ETE *if and only if it is welfare equivalent to a cross-monotonic mechanism with equal cost-shares.*[8]

This result is especially compelling when dividing costs that are symmetric. The subclass of cross-monotonic mechanism with equal cost-shares are the only GSP mechanisms meeting the basic equity requirement of ETE. This proposition rules out sequential mechanisms and also those GSP mechanisms discussed by Juarez[2007b] and Roughgarden[2007].

The downside of this proposition is that many interesting applications have cost functions that are not symmetric (see for instance Roughgarden[2007]), where ETE does not make sense.

6.3 Limited number of goods

When a social planner or seller has (can produce) less than n units of good, it is impossible to meet simultaneously ETE and GSP.[9] This is easy to check by looking at the utility profiles of the form $(x, \ldots, x)$, $x > 0$. By ETE, $G(x, \ldots, x) = \emptyset$ for all x. Hence, by proposition 2 above and taking into account that the smallest cost share in a cross-monotonic mechanism is achieved when serving N, the mechanism should not allocate any unit at all.

Moreover, when there is scarcity of the good, cross-monotonic mechanisms exclude ex-ante some agents from the mechanism.[10] That is, if only k units of good are available, $k < n$, then any cross-monotonic mechanism is such that $n - k$ agents are not served at any profile. To see this, notice coalition N never gets service, therefore the cost shares of N should have at least one coordinate equal to ∞. Thus the agent i with such coordinate never participates in the game because his smallest payment is achieved when serving N. We remove this agent from the game and proceed similarly with the remaining coalition $N \setminus i$, until we have removed at least $n - k$ agents.

On the other hand, there are many sequential mechanisms that do not ex-ante exclude any agent. If $k \geq 2$, some easy combination of sequential and cross-monotonic mechanisms can be constructed.

Definition 13 *Given an arbitrary order of the agents* $i_1, \ldots, i_n$ *and arbitrary prices (some of them may be infinity)* $x^1, x^2, \ldots, x^n$, *we define a priority mechanism as follows: Start with agent* i_1 *and offer him a unit of good at price* x^1. *If he accepts the offer then the mechanism*

[8] A cross-monotonic mechanism with equal cost-shares is a cross-monotonic set of cost-shares that allocate the same payments at every set of cost-shares. That is, the cost-shares x^S of the agents in S are such that $x^S_i = x^S_j$ for all $i, j \in S$.

[9] Except by the trivial mechanism that does not serve anyone at any profile.

[10] We say a mechanism *does not exclude ex-ante any agent* if for every agent i there is a utility profile u^i such that this agent is served.

stops there. If he does not accept the offer, then continue with agent i_2 and offer him a unit of good at price x^2. Continue similarly until some agent accepts the offer or we offered a unit to all agents.

Notice priority mechanisms are feasible sequential mechanisms for the feasible sequential tree such that agents are ordered linearly following the order $i_1, \ldots, i_n$; only the most leftist branch of the tree has prices equal to $(x^1, x^2, \ldots, x^n)$ and any other node has a price equal to ∞.

Proposition 3 *Suppose a mechanism is GSP, allocates at most one unit of good at any utility profile and does not exclude ex-ante any agent, then the mechanism is welfare equivalent to a priority mechanism.*

Notice this proposition is independent of the tie-breaking rule. In particular, it shows that when there is only one unit of good, a subclass of the feasible sequential mechanisms are the only GSP mechanisms that do not exclude ex-ante any agent.

The priority mechanisms are especially compelling when randomizations are disallowed. Give priority to the agents is natural in multiple settings. For instance, when allocating a scarce drug to sick people, priority is often given to the sicker people.

If randomization are allowed, many more interesting mechanisms emerge. For instance, the mechanism that allocates each object to the agents for free with probability $\frac{1}{n}$ is GSP and always allocates the object. This mechanism also satisfy ETE.

6.4 Feasible cost-sharing mechanisms

A cost-sharing function is a non-negative function $C : 2^N \to \mathbb{R}_+$ such that $C(S) \leq C(T)$ for $S \subseteq T$. It specifies the cost of serving every coalition of agents.

We say a mechanism (G, φ) is *feasible* for the cost-sharing function C if $\sum_{i \in G(u)} \varphi_i(G(u)) \geq C(G(u))$ for all utility profiles u. A feasible mechanism collects at least the cost of serving the agents $G(u)$ at every utility profile u.

A mechanism is *budget-balanced* if the cost is exactly collected at every utility profile u. That is, $\sum_{i \in G(u)} \varphi_i(G(u)) = C(G(u))$ for all u.

Given an arbitrary cost function, there exists sequential and cross-monotonic mechanism that are feasible. Indeed, any large enough homothetic expansion of the set of payments would lead to a feasible mechanism. For instance, for the cross-monotonic set of cost-shares, χ, consider the cross monotonic set of cost-shares $\lambda \cdot \chi = \{\lambda \cdot x^S | x^S \in \chi\}$ for some $\lambda > 0$. While we can always find a λ that makes the set of cost-shares $\lambda \cdot \chi$ feasible for any cost function C, such mechanism could be very wasteful (charge the agents too much).

Recent literature deals with the role of wastefulness. In particular, the companion paper Juarez[10] characterizes optimal mechanisms (using the worse absolute surplus loss) for an arbitrary *symmetric* cost function. When the cost function has decreasing average cost, Theorem 1 in Juarez[2011] shows that the optimal GSP mechanism would be the cross-monotonic mechanism where $x_i^S = AC(S)$ for all $i \in S$.[11] On the other hand, the optimal mechanism for a cost function with decreasing marginal cost would be a sequential mechanism.

6.4.1 Cross-monotonic mechanisms

Moulin[1999] shows that in the space of submodular cost functions, any mechanism that is budget-balanced, GSP and satisfies a strong consumer sovereignty condition should be implemented as a cross-monotonic mechanism for a set of cross-monotonic and budget-balanced cost shares. The result proposed by Theorem 1 is more general. We show that cross-monotonic mechanisms emerge simply from the combination of GSP and MAX. However, as shown in example 7, this does not imply the cost sharing function defined by $C(S) = \sum_{i \in S} x_i^S$ is submodular. Hence we capture Moulin's mechanisms and a few more.

Example 7 *Consider any cost function $C : 2^N \to \mathbb{R}_+$ such that its average cost function AC, $AC(S) = \frac{C(S)}{|S|}$, is not increasing as coalition increases.*

$x_i^S = AC(S)$ if $i \in S$, $x_i^S = 0$ if $i \notin S$, defines a cross-monotonic set of cost shares that covers the cost exactly.

It is easy to see that the monotonicity of AC does not imply the submodularity of C. Hence, there are cross-monotonic set of cost shares whose associated cost function is not concave.

In the space of cross-monotonic set of cost shares under equal-sharing,[12] the cost function generated by the mechanism is such that the average cost function AC is not increasing.

The problem of finding the cost function generated by an arbitrary cross-monotonic set of cost-shares is a difficult problem. Sprumont[1990] and Norde et al.[2002] provide characterizations of these cost functions in simple cases.

6.4.2 Sequential mechanisms

Sequential mechanisms are related to the incremental cost mechanisms of Moulin[1999]. That is, consider a supermodular (convex) cost function and a sequential tree. Start with the agent i_1 in the root and offer him a unit of good at price $C(i_1)$. If he buys, continue with the agent i_2 on the right of the tree and offer him a unit of good at price $C(i_1, i_2) - C(i_1)$. If i_1 did not buy, then offer the agent on the left of the tree, k_2, a unit of good at price $C(k_2)$. Proceed similarly with the following agents until you reach the end of the tree.

Theorem 1 in Moulin[1999] suggests that incremental cost mechanisms are GSP mechanisms when the cost function is supermodular. However, this is not true, as shown on the next example.

[11] This is similar to the main result of Moulin and Shenker[2001], except that Juarez[2011] allows more general mechanisms that can generate a budget surplus whereas Moulin and Shenker[2001] only work in the class of budget-balanced mechanisms.

[12] Under equal-sharing, all agents who are getting served pay the same price. In the class of GSP mechanisms, this property is weaker than ETE.

Example 8 *Consider the supermodular cost function:*

$$C(i) = 1, C(1,2) = 3, C(1,3) = 5, C(2,3) = 6, C(1,2,3) = 15.$$

By choosing the ordering $1 \succ 2 \succ 3$*, the cost shares are as follows:*
$x^{\{1,2,3\}} = (1,2,12), x^{\{1,2\}} = (1,2,0), x^{\{1,3\}} = (1,0,4), x^{\{2,3\}} = (0,1,5), x^{\{i\}} = 1_i$.

When the utility profile is $u = (1, 1.5, 4.5)$ *there are two options depending on whether 1 decides to get or not get a unit. If agent 1 gets a unit, then 2 does not get a unit and 3 gets a unit. Thus* $\{1,3\}$ *is served and the cost shares are* $(1,0,4)$*. If agent 1 does not get a unit, then 3 does not get a unit. Thus* $\{2\}$ *is served and the cost shares are* $(0,1,0)$*. Given that 1 is indifferent between getting and not getting a unit, he may help 2 or 3. Thus the mechanism cannot be* GSP*.*

What is important from Moulin[1999] is that incremental cost mechanism may not be fully GSP, but they are GSP except when agents are indifferent between getting and not getting a unit of good. Thus the mistake is very tiny.

Whenever the supermodular cost function and the ordering of the agents give a sequential mechanism that is feasible, it must be captured by a sequential mechanism discussed above.

On the other hand, given a feasible sequential mechanism, the associated budget balance cost function (the cost of S defined as the sum of the payments when S is served) may not be supermodular. Therefore, feasible sequential mechanisms capture even more mechanism than those generated by the incremental cost mechanisms.

Given the difficulty to describe the class of feasible sequential mechanisms, it is nearly impossible to describe the class of cost functions that are generated by arbitrary feasible sequential mechanisms. The exception to this difficulty comes when we restrict the attention to feasible sequential mechanisms under equal-sharing. The typical cost functions (up to renaming the agents) can be described by:

$$C(S) = \mid S \mid \max_{k \in S} a_k \text{ for } a_1 \geq a_2 \geq \cdots \geq a_n \geq 0. \tag{1}$$

Notice even these types of cost functions might not be supermodular. Indeed, we can easily find values such that $C(1,3) + C(2,3) = 2a_1 + 2a_2 > 3a_1 + a_3 = C(1,2,3) + C(3)$.

The feasible sequential mechanism for this cost function is shown in figure 6 for five agents.

7 Conclusions

This paper characterizes the GSP mechanisms under two alternative continuity conditions. On one hand, cross-monotonic mechanisms are characterized by GSP and MAX. These mechanisms are very useful when symmetry is required. However, they are very inefficient when there is scarcity of the good.

On the other hand, sequential mechanisms are characterized by GSP and MIN. These mechanisms are appropriate when there is scarcity of the good, for instance when there is

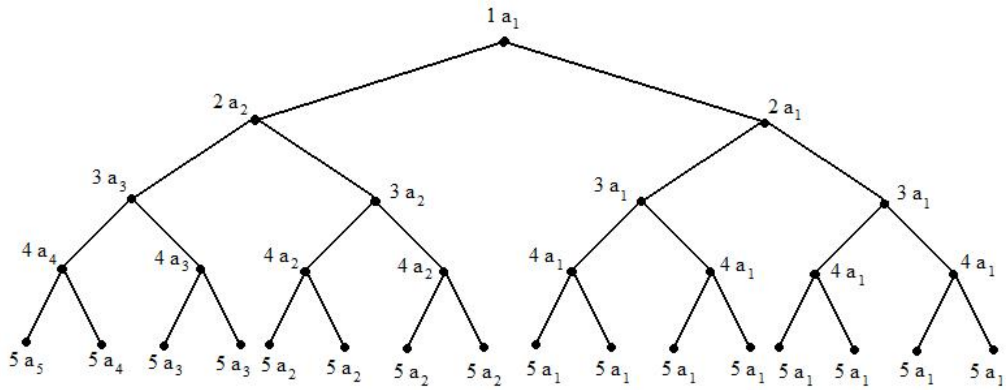

Figure 6: Five agents feasible sequential mechanism under equal-sharing.

only one unit of good available. Unfortunately, deterministic sequential mechanisms fail standard equity such as equal treatment of equals.

Group strategyproof mechanisms without any of the two alternative continuity conditions can be easily constructed, for instance some priority compositions of sequential an cross monotonic mechanism are GSP (see Juarez[2007b] or Roughgarden[2007]). However, the full characterization of GSP mechanisms in this economy is an open question.

Proofs

Proof of Theorem 2.

Feasible sequential mechanisms meet MIN and GSP.

Feasible sequential mechanisms trivially meet MIN.

We prove by contradiction these mechanisms meet GSP. Assume coalition $\widetilde{S}$ profitably misreports $\widetilde{u}_{\widetilde{S}}$ at the true profile u. Let $k \in \widetilde{S}$ be an agent who strictly increases his net utility by misreporting. Let ζ and ζ' be the nodes that contain agent k in the paths that generate $G(u)$ and $G(\widetilde{u}_{\widetilde{S}}, u_{-\widetilde{S}})$ respectively.

First notice ζ is on the left of ζ'. To see this, let i^* be the agent in the terminal node of $P_0(\zeta) \sqcap P_0(\zeta')$. Then, in order to move from $P_0(\zeta)$ to $P_0(\zeta')$, agent i^* misreports. If i^* is winning in $P_0(\zeta)$ then by MIN his net utility is positive, so he will never agree to move to $P_0(\zeta')$ because he is not served there.

Let L and R be as in definition 9. Since agent k strictly increases his net utility, then $x_k^L > x_k^R$. Assume condition (a) of feasibility is satisfied. That is, there exist nodes $\widetilde{\zeta}$ and $\bar{\zeta}$ that contain the same agent i such that $\widetilde{\zeta}$ is losing in L, $\bar{\zeta}$ is winning in R and $x_i^L < x_i^R$. Since $\widetilde{\zeta}$ is losing in L then $u_i \leq x_i^L < x_i^R$. Thus, for the path $P_0(\zeta')$ to realize, $i \in \widetilde{S}$ and $\widetilde{u}_i > x_i^R$. Hence the net utility of agent i is negative when he misreports because $u_i < x_i^R$. This is a contradiction.

On the other hand, assume condition (b) of feasibility is satisfied. That is, there exist nodes $\widetilde{\zeta}$ and $\bar{\zeta}$ that contain the same agent i such that $\widetilde{\zeta}$ is winning in L, $\bar{\zeta}$ is losing in R and $x_i^L \geq x_i^R$. Given that $\widetilde{\zeta}$ is winning in L, $u_i > x_i^L \geq x_i^R$. Thus, for the path $P_0(\zeta')$ to realize, $i \in \widetilde{S}$ and $\widetilde{u}_i \leq x_i^R$. Hence, the net utility of agent i strictly decreases from $u_i - x_i^L$ to zero when he misreports. This is a contradiction.

Any GSP and MIN mechanism is a feasible sequential mechanism.

Let (G, φ) a mechanism that meets GSP and MIN. Steps 1, 2 and 3 are three preliminary properties of (G, φ). Steps 4 and 5 prove (G, φ) is a sequential mechanism. Step 6 proves it is a feasible sequential mechanism.

Step 1. If $G(u) = S^*$ and $\varphi(u) = \varphi^*$, then for all $\widetilde{u}$ such that $\widetilde{u}_{S^*} >> \varphi_{S^*}$ and $\widetilde{u}_{N\setminus S^*} \leq u_{N\setminus S^*}$, $G(\widetilde{u}) = S^*$ and $\varphi(\widetilde{u}) = \varphi^*$.

Proof.

First notice that by MIN, an agent gets positive net utility if and only if he is served.

Let $i \in S^*$. Then $G(\widetilde{u}_i, u_{-i}) = S^*$ and $\varphi(\widetilde{u}_i, u_{-i}) = \varphi^*$. To see this, if $i \notin G(\widetilde{u}_i, u_{-i})$ or $\varphi_i(\widetilde{u}_i, u_{-i}) > \varphi_i^*$, then agent i misreports u_i when the true profile is $(\widetilde{u}_i, u_{-i})$, which contradicts SP. On the other hand, if $i \in G(\widetilde{u}_i, u_{-i})$ and $\varphi_i(\widetilde{u}_i, u_{-i}) < \varphi_i^*$, then agent i misreports $\widetilde{u}_i$ when the true profile is u, which also contradicts SP. Therefore, $i \in G(\widetilde{u}_i, u_{-i})$ and $\varphi_i(\widetilde{u}_i, u_{-i}) = \varphi_i^*$.

Let j, $j \neq i$. If $NU_j(\widetilde{u}_i, u_{-i}) > NU_j(u)$, then agent i helps j by misreporting $\widetilde{u}_i$ when the true profile is u. This contradicts GSP. The case $NU_j(\widetilde{u}_i, u_{-i}) < NU_j(u)$ is analogous. Thus $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u)$ for all $j \neq i$. Therefore, by MIN $G(\widetilde{u}_i, u_{-i}) = S^*$ and $\varphi(\widetilde{u}_i, u_{-i}) = \varphi^*$.

By applying the previous argument to each agent in S^*, we have that $G(\widetilde{u}_{S^*}, u_{-S^*}) = S^*$ and $\varphi(\widetilde{u}_{S^*}, u_{-S^*}) = \varphi^*$.

Let $j \notin S^*$. Then $G(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = S^*$ and $\varphi(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = \varphi^*$. First notice that $j \notin G(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j})$, otherwise by voluntary participation

$$\varphi_j(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) < \widetilde{u}_j \leq u_j.$$

Thus agent j misreports $\widetilde{u}_j$ when true profile is $(\widetilde{u}_{S^*}, u_{-S^*})$. This contradicts SP.

On the other hand, if $NU_k(\widetilde{u}_{S^* \cup j}, u_{-(S^* \cup j)}) < NU_k(\widetilde{u}_{S^*}, u_{-S^*})$ for some $k \neq j$, then agent j helps k by reporting $\widetilde{u}_j$ when true profile is $(\widetilde{u}_{S^*}, u_{-S^*})$, this contradicts GSP. Similarly, by GSP $NU_k(\widetilde{u}_{S^* \cup j}, u_{-(S^* \cup j)}) > NU_k(\widetilde{u}_{S^*}, u_{-S^*})$ cannot occur. Thus $NU_k(\widetilde{u}_{S^* \cup j}, u_{-(S^* \cup j)}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*})$ for all $k \neq j$. Hence, by MIN $G(\widetilde{u}_{S^* \cup j}, u_{-(S^* \cup j)}) = S^*$ and $\varphi(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = \varphi^*$.

By repeatedly using the previous argument to every agent in $N \setminus S^*$, we have that $G(\widetilde{u}) = S^*$ and $\varphi(\widetilde{u}) = \varphi^*$.

Step 2. If $G(u) = G(\widetilde{u})$ then $\varphi(u) = \varphi(\widetilde{u})$.

Proof.

Let $S^* = G(u) = G(\widetilde{u})$, $\bar{v}_S = \max(\widetilde{u}_S, u_S)$ and $\bar{v}_{N \setminus S} = \min(\widetilde{u}_{N \setminus S}, u_{N \setminus S})$ (where max and min are taken coordinate by coordinate).

By step 1, comparing $\bar{v}$ and u, $G(\bar{v}) = S^*$ and $\varphi(\bar{v}) = \varphi(u)$. Similarly, comparing $\bar{v}$ and $\widetilde{u}$, $\varphi(\bar{v}) = \varphi(\widetilde{u})$.

By step 2, there exist at most one vector of payments for every coalition. Let x^{S^*} be the payment of coalition S^* when S^* is served at some profile.

Step 3. Let u be such that $G(u) = S^*$ and $\varphi(u) = \varphi^*$. Then for every $i \in S^*$ and $u_i^* \leq \varphi_i^*$, $S^* \backslash i \subseteq G(u_i^*, u_{-i})$ and $\varphi_{S^* \backslash i}(u_i^*, u_{-i}) \leq \varphi^*_{S^* \backslash i}$.

Proof.

First notice that for every $j \in S^* \setminus i^*$, $j \in G(\varphi_i^*, u_{-i})$ and $\varphi_j(\varphi_i^*, u_{-i}) \leq \varphi_j^*$. Indeed, by MIN the net utility of agent j at u is positive. If $j \notin G(\varphi_i^*, u_{-i})$ or $\varphi_j(\varphi_i^*, u_{-i}) > \varphi_j^*$ then agent i can help j by misreporting u_i when the true profile is (φ_i^*, u_{-i}) : By MIN, agent i is not being served at the profile (φ_i^*, u_{-i}), thus he is indifferent between misreporting u_i and getting a unit at price φ_i^*, or truly reporting φ_i^* and not getting a unit, whereby agent j is better of at u. This contradicts GSP.

Finally, since $i \notin G(\varphi_i^*, u_{-i})$ and by step 1, $G(u_i^*, u_{-i}) = G(\varphi_i^*, u_{-i})$ and $\varphi(u_i^*, u_{-i}) = \varphi(\varphi_i^*, u_{-i})$ for all $u_i^* \leq \varphi_i^*$.

Step 3.1 If $G(u) = S^*$, then for any T, $T \subset S^*$, there exist $\widetilde{u}$ such that $G(\widetilde{u}) = T$ and $x_T^T \leq x_T^{S^*}$.

Proof.

Let $\bar{u} = (u_{S^*}, 0_{-S^*})$. By step 1, $G(\bar{u}) = S^*$ and $\varphi(\bar{u}) = x^{S^*}$. Let $i \in S^*$. By MIN $i \notin G(x_i^{S^*}, \bar{u}_{-i})$. By step 3, $S^* \setminus i \subset G(x_i^{S^*}, \bar{u}_{-i})$. Since the utilities of agents outside S^* are zero, then by MIN $S^* \setminus i = G(x_i^{S^*}, \bar{u}_{-i})$. Thus by step 3, $x_{S\setminus i^*}^{S\setminus i^*} \leq x_{S\setminus i^*}^{S}$. Finally, to check the claim we repeatedly apply the above argument to every agent in $S^* \setminus T$.

Step 4. Assume there is u^* such that $G(u^*) = N$. Then, there is an agent to whom is offered a unit of good at a price that is independent of the utilities of the other agents (we say this agent has priority).

We prove this by induction in the size of N.

If $N = \{1\}$ then the GSP and MIN mechanisms are clearly fixed cost mechanisms. That is, there is a fixed price x, $x \in [0, \infty]$ such that if $u_1 > x$ then 1 is served at price x. If $u_1 \leq x$ then he is not served.

For the induction hypothesis, assume that for any GSP and MIN mechanism for $n-1$ agents there is an agent who has priority. Let (G, φ) be a mechanism for the agents in $N = \{1, \ldots, n\}$.

For every j, consider the utility profiles where agent j has zero utility, that is

$$U^j = \{u \in \mathbb{R}_+^N \mid u_j = 0\}.$$

By MIN, agent j is not being served at any profile of U^j. Thus, the restriction of (G, φ) to U^j defines a MIN and GSP mechanism for the agents in $N \setminus j$. Let $\rho^j = \{x^S \mid j \notin S\}$ be the set of payments in this mechanism. Notice because N is being served, then by step 3.1 every coalition $S \subset N$ is being served. In particular ρ^j contains a payment for every group of agents that does not contain agent j. Also, notice that by step 2 if $x^T \in \rho^j$ and $\tilde{x}^T \in \rho^k$ then $x^T = \tilde{x}^T$.

Finally by step 3.1 payments are nondecreasing as coalition increases. That is, if $S \subset T$ then $x_S^S \leq x_S^T$.

By the induction hypothesis, on ρ^1 there is an agent i_1 who has priority. The monotonicity of the payments implies $x_{i_1}^{N\setminus 1} = x_{i_1}^{i_1}$. Similarly, there is an agent who has priority on ρ^{i_1}. Call this agent i_2, thus $x_{i_2}^{N\setminus i_1} = x_{i_2}^{i_2}$. We continue this procedure until we reach a cycle. Without loss of generality, we assume the cycle is $i_1, i_2, \ldots, i_k$. This means i_{j+1} has priority on ρ^{i_j} for $j = 1, \ldots, k-1$, and i_1 has priority on ρ^{i_k}.

Case 1. The cycle has size less than n, that is $k < n$.

Let $\bar{v}_{N\setminus\{i_1,i_2,\ldots,i_k\}}$ be such that $\bar{v}_{N\setminus\{i_1,i_2,\ldots,i_k\}} >> x_{N\setminus\{i_1,i_2,\ldots,i_k\}}^N$.

Consider the profiles

$$U = \{u \in \mathbb{R}_+^N \mid u_{N\setminus\{i_1,i_2,\ldots,i_k\}} = \bar{v}_{N\setminus\{i_1,i_2,\ldots,i_k\}}\}.$$

Notice that for every $u \in U$, $N\setminus\{i_1, i_2, \ldots, i_k\} \subset G(u)$. Indeed, consider $(\widetilde{u}_{\{i_1,i_2,\ldots,i_k\}}, u_{-\{i_1,i_2,\ldots,i_k\}})$ such that $\widetilde{u}_{\{i_1,i_2,\ldots,i_k\}} >> x_{i_1,i_2,\ldots,i_k}^N$. By step 1, $G(\widetilde{u}_{\{i_1,i_2,\ldots,i_k\}}, u_{-\{i_1,i_2,\ldots,i_k\}}) = N$. By steps 1 and

3, $N\backslash\{i_1\} \subseteq G(u_{i_1}, \widetilde{u}_{\{i_2,\ldots,i_k\}}, u_{-\{i_1,i_2,\ldots,i_k\}})$. Similarly, $N\backslash\{i_1, i_2\} \subseteq G(u_{i_1,i_2}, \widetilde{u}_{\{i_3,\ldots,i_k\}}, u_{-\{i_1,i_2,\ldots,i_k\}})$. Continuing this way, $N \setminus \{i_1, i_2, \ldots, i_k\} \subseteq G(u_{\{i_1,i_2,\ldots,i_k\}}, u_{-\{i_1,i_2,\ldots,i_k\}})$.

By step 3.1, for every coalition T such that $N \setminus \{i_1, i_2, \ldots, i_k\} \subset T$, there is $\widetilde{u} \in U$ such that $G(\widetilde{u}) = T$. This is clear because coalition N is being served at some profile of U, so we can reduce (one agent at a time) the utility of the agents not in T to zero.

Clearly, the mechanism restricted to U defines a GSP mechanism for the agents in $\{i_1, i_2, \ldots, i_k\}$. By the induction hypothesis, there is an agent who has priority, say i_1. Thus, $x_{i_1}^{N\backslash\{i_2,\ldots,i_k\}} = x_{i_1}^N$. On the other hand, because i_1 has priority on ρ^{i_k}, $x_{i_1}^{i_1} = x_{i_1}^{N\backslash\{i_2,\ldots,i_k\}}$. Therefore, $x_{i_1}^N = x_{i_1}^{i_1}$. Hence by the monotonicity of the payments $x_{i_1}^T = x_{i_1}^S$ for all $S, T \subseteq N$ such that $i_1 \in S, T$.

Finally, we prove agent i_1 has priority. Assume there is u such that $u_{i_1} > x_{i_1}^{i_1}$ but $i_1 \notin G(u)$. Consider the profile $(u_{i_1}, \tilde{u}_{-i_1})$ where $\tilde{u}_{-i_1} >> \max(x_{-i_1}^N, u_{-i_1})$ and max is taken coordinate by coordinate. By step 1, $G(u_{i_1}, \tilde{u}_{-i_1}) = N$. By step 1 and 3, $N \setminus \{i_2\} \subseteq G(u_{i_1}, u_{i_2}, \tilde{u}_{-i_1,i_2})$. Similarly, by steps 1 and 3, $N \setminus \{i_2, i_3\} \subseteq G(u_{i_1}, u_{i_2}, u_{i_3}, \tilde{u}_{-i_1,i_2,i_3})$. Continuing this way, $\{i_1\} \subseteq G(u)$. This is a contradiction.

Case 2. The cycle has size n, that is $k = n$.

Without loss of generality, assume agent 2 has priority over $N \setminus 1$, agent 3 has priority over $N \setminus 2, \ldots,$ etc. Thus,

$$x_2^2 = x_2^{N\backslash 1}, \ldots x_3^3 = x_3^{N\backslash 2}, \ldots x_1^1 = x_1^{N\backslash n}. \tag{2}$$

Also, assume to get a contradiction, that there is no agent who has priority. That is,

$$x_2^{N\backslash 1} < x_2^N, x_3^{N\backslash 2} < x_3^N, \ldots, x_1^{N\backslash n} < x_1^N.$$

Let u^* be such that $G(u^*) = N$. By MIN, $u^* >> x^N$.

By step 3, $2 \in G(x_1^N, u_{-1}^*)$ and 2 pays x_2^2, $x_2^2 < x_2^N$, because he has priority on ρ^1. Also by step 3, $2 \in G(x_{1,3}^N, u_{-1,3}^*)$ and 2 pays not more than x_2^2. Continuing similarly, $2 \in G(x_{-2}^N, u_2^*)$ and 2 pays not more than x_2^2. By step 1, $2 \in G(x^N)$ because $u_2^* > x_2^N > x_2^2$.

Finally, since everything is symmetric, $G(x^N) = N$. This contradicts MIN.

Step 5. Assume there is no u such that $G(u) = N$. If the mechanism is not trivial ($G(u) \neq \emptyset$ for some u), there is an agent who has finite priority. That is, there is an agent i^* and a payment x^*, $0 \leq x^* < \infty$, such that $i^* \in G(u)$ for all u such that $u_{i^*} > x^*$.

First notice there is a group of agents S^* who has priority. That is, for all $\widetilde{u}$ such that $\widetilde{u}_{S^*} \geq x_{S^*}^{S^*}$, $G(\widetilde{u}) = S^*$. To see this, consider $\widetilde{u}$ such that $\widetilde{u} >> x^T$ for all possible payments x^T, $x^T = \varphi(v)$ for some v (we know by step 2 that there is at most one vector of payments for every coalition, thus it is feasible to choose such $\widetilde{u}$). Let S^* be such that $G(\widetilde{u}) = S^*$. Notice that, for any i, $i \notin S^*$, $G(\widetilde{u}_{-i}, \bar{v}_i) = S^*$ for all $\bar{v}_i$. Indeed, if $\bar{v}_i \leq \widetilde{u}_i$ then by step 1 $i \notin G(\widetilde{u}_{-i}, \bar{v}_i)$. On the other hand, if $\bar{v}_i > \widetilde{u}_i$, then $i \notin G(\widetilde{u}_{-i}, \bar{v}_i)$. This is easy to see by contradiction, assume $i \in G(\widetilde{u}_{-i}, \bar{v}_i)$, then by the choice of $\widetilde{u}$, $\varphi_i(\widetilde{u}_{-i}, \bar{v}_i) < \widetilde{u}_i < \bar{v}_i$. Therefore, by step 1, $i \in G(\widetilde{u})$, which is a contradiction.

Hence, $G(\widetilde{u}_{-i}, \bar{v}_i) = S^*$ for all $\bar{v}_i$. Thus, by changing the utilities of the agents in $N \setminus S^*$ one at a time, $G(\widetilde{u}_{S^*}, u_{-S^*}) = S^*$. Hence by step 1, $G(\bar{u}_{S^*}, u_{-S^*}) = S^*$ for all $\bar{u}_{S^*} \geq x^{S^*}_{S^*}$ and all u_{-S^*}.

We now prove step 5 by induction. For $n = 1$, if $G(u) \neq 1$ for all u then clearly the mechanism is trivial ($G(u) = \emptyset$ for all u). So the claim is true.

For the induction hypothesis, assume the claim is true for any mechanism of $n-1$ agents. We prove it for any mechanism of n agents.

Let S^* be defined as above and $j \notin S^*$. Consider the restriction of the mechanism to $U^j = \{u \in \mathbb{R}^N_+ \mid u_j = 0\}$. Then this restriction is a GSP and MIN mechanism for the agents in $N \setminus j$. By induction and step 4, there is an agent i^* who has (finite) priority for the agents $N \setminus j$. Clearly $i^* \in S^*$, otherwise his payment is dependent on the agents in S^*.

We now prove by contradiction that for any profile u_{-i^*}, i^* has priority. Assume there is u such that $f_{i^*}(u_{-i^*}) \neq x^{S^*}_{i^*}$, where $f_{i^*}(u_{-i^*})$ is the price of a unit of good that the mechanism makes to agent i^* when the utilities of the other agents are u_{-i^*} (recall this function exists because the mechanism meets SP, VP and NNT). Let $u_{i^*} = \tilde{u}_{i^*}$, a utility bigger than all possible payments for agent i^*, in particular $u_{i^*} > x^{S^*}_{i^*}$. First notice that $j \in G(u)$, otherwise, by step 1 $G(u) = G(0, u_{-j})$ and $\varphi(u) = \varphi(0, u_{-j})$. Thus i^* is served at u at a price equal to $x^{S^*}_{i^*}$, which contradictions our assumptions. Hence $j \in G(u)$. By step 3, $f_{i^*}(u_{-i^*}) > x^{S^*}_{i^*}$.

Let $k \in S^* \setminus i^*$ and $\bar{u}_k > \max(u_k, x^{S^*}_k)$, then $f_{i^*}(\bar{u}_k, u_{-k,i^*}) \geq x^{S^*}_{i^*}$. Indeed, if $k \in G(u)$, then by step 1 $G(\bar{u}_k, u_{-k}) = G(u)$ and $f_{i^*}(\bar{u}_k, u_{-k,i^*}) = f_{i^*}(u_k, u_{-k,i^*}) > x^{S^*}_{i^*}$. On the other hand, if $k \notin G(u)$ and $k \notin G(\bar{u}_k, u_{-k})$, then by step 1 $G(\bar{u}_k, u_{-k}) = G(u)$ and $f_{i^*}(\bar{u}_k, u_{-k,i^*}) = f_{i^*}(u_k, u_{-k,i^*}) > x^{S^*}_{i^*}$. Finally, if $k \notin G(u)$ and $k \in G(\bar{u}_k, u_{-k})$, then by step 3 $f_{i^*}(\bar{u}_k, u_{-i^*,k}) \geq f_{i^*}(u_k, u_{-i^*,k}) > x^{S^*}_{i^*}$.

By repeatedly using the above argument to every agent in $S^* \setminus i^*$ we conclude that $f_{i^*}(u_{-S^*}, \bar{u}_{S^* \setminus i^*}) > x^{S^*}_{i^*}$ for some $(u_{i^*}, \bar{u}_{S^* \setminus i^*}) \geq x^{S^*}_{S^*}$. This contradicts the priority of coalition S^*.

Steps 4 and 5 showed that for any GSP and MIN mechanism there exists an agent whose payment is independent of the other agent's utilities. By induction, this clearly implies the mechanism is sequential.

Step 6. The mechanism is implemented by a feasible sequential tree.

Proof.

Given a sequential mechanism (definition 7) that meets GSP and MIN, we show by contradiction that this mechanism is feasible (as in definition 9).

Assume the sequential tree that implements this mechanism is not feasible. Let ζ and ζ' be two achievable[13] nodes that contain the same agent k such that $x^L_k > x^R_k$, and for every two nodes $\tilde{\zeta} \in L$ and $\bar{\zeta} \in R$ that contains the same agent i, one of the next conditions hold:

1. $\tilde{\zeta}$ is losing in L, $\bar{\zeta}$ is winning in R and $x^L_i \geq x^R_i$.
2. $\tilde{\zeta}$ is winning in L and $\bar{\zeta}$ is losing in R and $x^L_i < x^R_i$.

[13]That is, all winning agents in their paths to the root of the tree have finite prices

3. $\tilde{\zeta}$ and $\bar{\zeta}$ are losing in L and R.

4. $\tilde{\zeta}$ and $\bar{\zeta}$ are winning in L and R.

Let i^* be the agent in the terminal node of $P_0(\zeta) \sqcap P_0(\zeta')$. Fix a utility profile u such that:

a. u_{i^*} equal the price of his node.

b. u_k such that $x_k^L > u_k > x_k^R$

c. $u_i = x_i^R$ if condition 1 holds.

d. $u_i = \frac{x_i^L + x_i^R}{2}$ if condition 2 holds.

e. $u_i = 0$ if condition 3 holds.

f. u_i such that $u_i > \max(x_i^L, x_i^R)$ if condition 4 holds.

g. If j is unique winning agent in $(P_0(\zeta) \sqcup P_0(\zeta')) \setminus (L \sqcap R)$ then u_j is bigger than the price of its node.

h. If j is unique losing agent in $(P_0(\zeta) \sqcup P_0(\zeta')) \setminus (L \sqcap R)$ then $u_j = 0$.

i. Any other agent has zero utility.

First notice the profile u realizes the path $P_0(\zeta)$.

If an agent is losing in $P_0(\zeta)$ then either his utility equals to zero, or condition 1 is satisfied, or he is i^*. If his utility equals to zero, by MIN he is not served. If condition 1 is satisfied then $u_i = x_i^R \leq x_i^L$ so he is not served. If he is i^*, then his utility equal the price of his node, so he is not served.

On the other hand, if an agent is winning in $P_0(\zeta)$ then his utility is bigger than the price of his node. To see this, if condition 2 is satisfied then by part d he is served. If condition 4 is satisfied, then by part f he is served. The remaining winning agents are served by part g.

Let T be the common agents who meet condition 1 and $S = T \cup \{i^*, k\}$. We now check that when the true profile is u, coalition S can profitably misreport. First notice all agents in S are not being served at u, so they get zero net utility.

Let $\widetilde{u}_S$ be such that:

- $\widetilde{u}_i > u_i$ if $i \in T \cup \{i^*\}$.
- $\widetilde{u}_k = u_k$

Then at the profile $(\widetilde{u}_S, u_{-S})$ the path $P_0(\zeta')$ realizes. Indeed, an agent j whose node is in $(P_0(\zeta) \sqcup P_0(\zeta')) \setminus (L \sqcap R)$ is obviously served if winning and not served of losing. If i meets condition 2, then $u_i = \frac{x_i^R + x_i^L}{2} < x_i^R$, so he is not served. If i meets condition 3, then by e his utility equals zero, thus he is not served. If i meets condition 4, then by f his utility is bigger than x_i^R, thus he is served. Also, i^* is winning and he is served at a price equal to his true valuation u_i, thus his net utility is zero. If $i \in T$, that is $i \in L \sqcap R$ is winning in R, then he is being served at a price equal to his valuation because $\widetilde{u}_i > u_i = x_i^R$, thus his net utility is zero. Finally, agent k is being served at a price x_k^R, $u_k > x_k^R$. Hence his net utility increases by misreporting.

Proof of Theorem 1.

Cross-monotonic mechanisms meet MAX and GSP.

Cross-monotonic mechanisms clearly meet MAX.

We prove by contradiction that these mechanisms meet GSP. Consider the cross-monotonic mechanism generated by the cross monotonic set of cost shares $\{x^S \mid S \subseteq N\}$.

Consider the offer function $f_i(u_{-i})$, the price agent i should pay to get a unit of good when the utilities of the remaining agents are u_{-i}. That is, $f_i(u_{-i}) = x_i^{S^*}$ where S^* is the maximal reachable coalition at (∞, u_{-i}). By cross-monotonicity of the cost shares and the definition of f_i, the offer function does not increase when u_{-i} increases. That is, if $v_{-i} \geq u_{-i}$ then $f_i(v_{-i}) \leq f_i(u_{-i})$.

Furthermore, the set of offer functions $f_1, \ldots, f_n$ generate precisely the mechanism (G, φ). That is, $G(u) = S^*$ if and only if $u_i \geq f_i(u_{-i})$ for all $i \in S^*$ and $u_j < f_j(u_{-j})$ for all $j \notin S^*$. Indeed, to prove the only if part, assume $G(u) = S^*$. Let $i \in S^*$, since S^* is the maximal reachable coalition at u, then by cross monotonicity S^* is the maximal reachable coalition at (∞, u_{-i}), thus $f_i(u_{-i}) = x_i^{S^*} \leq u_i$. Let $j \notin S^*$ and T the maximal reachable coalition at (∞, u_{-j}). To get a contradiction, assume that $u_j \geq x_j^T = f_j(u_{-j})$. Then T is reachable at u, thus $T \subseteq S^*$. Furthermore, since $j \in T$, then $j \in S^*$, which is a contradiction. We now prove the if part. Let T be the maximal reachable coalition at (∞, u_{-i}) and assume $u_i \geq f_i(u_{-i}) = x_i^T$. Then $i \in T$ and T is reachable at u. Thus $T \subseteq G(u)$, hence $i \in G(u)$. On the other hand, let $\widetilde{T}$ be the maximal reachable coalition at (∞, u_{-j}) and assume $u_j < f_j(u_{-j}) = x_j^{\widetilde{T}}$. To get a contradiction, assume that $j \in G(u)$. Then by monotonicity, $G(u) \subseteq \widetilde{T}$. Thus $u_j \geq x_j^{G(u)} \geq x_j^{\widetilde{T}}$, which contradicts our initial assumptions.

Assume coalition $\widetilde{S}$ profitably misreports $\widetilde{u}_{\widetilde{S}}$ when the true profile is u. Let $\bar{v}_{\widetilde{S}} = \max(u_{\widetilde{S}}, \widetilde{u}_{\widetilde{S}})$, where max is taken coordinate by coordinate. Because the offer function does not increase, coalition $\widetilde{S}$ also profits from misreporting $\bar{v}_{\widetilde{S}}$ when the true profile is u.

By monotonicity of the offer function, $G(u) \subseteq G(\bar{v}_{\widetilde{S}}, u_{-\widetilde{S}})$. Since coalition $\widetilde{S}$ profits from misreporting, then $G(u) \subsetneq G(\bar{v}_{\widetilde{S}}, u_{-\widetilde{S}})$. Since $G(\bar{v}_{\widetilde{S}}, u_{-\widetilde{S}})$ is not reachable at u, then there is an agent i such that $u_i < x_i^{G(\bar{v}_{\widetilde{S}}, u_{-\widetilde{S}})}$. Clearly $i \notin \widetilde{S}$ would contradict voluntary participation. Thus $i \in \widetilde{S}$, hence i is worse off by misreporting, which is a contradiction.

Any mechanism that is MAX and GSP is cross-monotonic.

Let (G, φ) be a mechanism that meets these properties. Recall that $f_i(u_{-i})$ is the price agent i should pay to get a unit of good when the utilities of the remaining agents are u_{-i}. Also, $NU_i(u)$ denotes the net utility of agent i at the profile u.

The proof of this part is divided in four steps. Steps 1 and 2 are very similar to step 1 and 2 in the proof of Theorem 2. However, step 1 involves more details because MAX does not imply that an agent is served if and only if his net utility is positive.

Step 0.[Monotonicity] $f_j(\widetilde{u}_i, u_{-ij}) \leq f_j(u_i, u_{-ij})$ for all $\widetilde{u}_i > u_i$.

Proof.

We prove this by contradiction. Suppose $f_j(\widetilde{u}_i, u_{-ij}) > f_j(u_i, u_{-ij})$. Let $\bar{v}_j$ be such that $f_j(\widetilde{u}_i, u_{-ij}) > \bar{v}_j > f_j(u_i, u_{-ij})$.

Case 1. $f_i(\bar{v}_j, u_{-ij}) > \widetilde{u}_i$.

By SP, agent i is not served at the profiles $(\widetilde{u}_i, \bar{v}_j, u_{-ij})$ and $(u_i, \bar{v}_j, u_{-ij})$ because $f_i(\bar{v}_j, u_{-ij}) > \widetilde{u}_i > u_i$. Hence when the true utility profile is $(\widetilde{u}_i, \bar{v}_j, u_{-ij})$, agent i can help j by misreporting u_i. This contradicts GSP.

Case 2. $f_i(\bar{v}_j, u_{-ij}) \leq u_i$.

By SP and MAX, agent i is served at the profiles $(\widetilde{u}_i, \bar{v}_j, u_{-ij})$ and $(u_i, \bar{v}_j, u_{-ij})$ because $f_i(\bar{v}_j, u_{-ij}) \leq u_i < \widetilde{u}_i$. Hence, similarly to case 1, when the true utility profile is $(\widetilde{u}_i, \bar{v}_j, u_{-ij})$, agent i can help j by misreporting u_i. This also contradicts GSP.

Case 3. $u_i < f_i(\bar{v}_j, u_{-ij}) \leq \widetilde{u}_i$.

Let $\widehat{u}_i = f_i(\bar{v}_j, u_{-ij})$. By SP and MAX, agent i is being served at price $\widehat{u}_i$ at the profiles $(\widetilde{u}_i, \bar{v}_j, u_{-ij})$ and $(\widehat{u}_i, \bar{v}_j, u_{-ij})$. Thus, by GSP $f_j(\widehat{u}_i, u_{-ij}) \geq \bar{v}_j$. To see this, assume $f_j(\widehat{u}_i, u_{-ij}) < \bar{v}_j$. Then, when the true profile is $(\widetilde{u}_i, \bar{v}_j, u_{-ij})$, agent i helps j by misreporting $\widehat{u}_i$. This contradicts GSP.

Hence, at the true profile $(\widehat{u}_i, \bar{v}_j, u_{-ij})$, agents i and j get zero net utility because $f_j(\widehat{u}_i, u_{-ij}) \geq \bar{v}_j$ and $\widehat{u}_i = f_i(\bar{v}_j, u_{-ij})$. Thus agent i helps j by reporting u_i : Agent i is not served at the misreport because $u_i < f_i(\bar{v}_j, u_{-ij})$, however agent j is better off because $\bar{v}_j > f_j(u_i, u_{-ij})$. This contradicts GSP.

Step 1. If $G(u) = S^*$ and $\varphi(u) = \varphi^*$ then for all $\widetilde{u}$ such that $\widetilde{u}_{S^*} \geq \varphi_{S^*}$ and $\widetilde{u}_{N \setminus S^*} \leq u_{N \setminus S^*}$, $G(\widetilde{u}) = S^*$ and $\varphi(\widetilde{u}) = \varphi^*$.

Proof.

We prove step 1 in steps 1.1 and 1.2.

Step 1.1. Let $i \in S^*$ and $\widetilde{u}_i \geq f_i(u_{-i}) = \varphi^*_i$. We will prove that $G(\widetilde{u}_i, u_{-i}) = S^*$ and $\varphi(\widetilde{u}_i, u_{-i}) = \varphi^*$.

First, notice that by SP and MAX, $i \in G(\widetilde{u}_i, u_{-i})$ and $\varphi_i(\widetilde{u}_i, u_{-i}) = \varphi^*_i$.

Second, notice $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u)$ for all $j \neq i$. To see this, if $NU_j(\widetilde{u}_i, u_{-i}) > NU_j(u)$, then when the true profile is u, agent i helps j by reporting $\widetilde{u}_i$. This contradicts GSP. Similarly, if $NU_j(\widetilde{u}_i, u_{-i}) < NU_j(u)$ then agent i helps j by misreporting $\widetilde{u}_i$ when the true utility profile is u.

Third, notice if $j \in S^* \setminus i$ and $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u) > 0$ then $j \in G(\widetilde{u}_i, u_{-i})$ and $\varphi_j(\widetilde{u}_i, u_{-i}) = \varphi^*_j$.

Finally, to get a contradiction, assume $G(\widetilde{u}_i, u_{-i}) \neq S^*$. Then, there is an agent j such that $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u) = 0$ and either (A.1.) $j \in S^*$ but $j \notin G(\widetilde{u}_i, u_{-i})$ or (A.2.) $j \notin S^*$ but $j \in G(\widetilde{u}_i, u_{-i})$. We show next that these situations cannot occur.

Case A.1. Assume $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u) = 0$, $j \in S^*$ but $j \notin G(\widetilde{u}_i, u_{-i})$.

Since $j \notin G(\widetilde{u}_i, u_{-i})$, by SP and MAX $f_j(\widetilde{u}_i, u_{-ij}) > u_j = f_j(u_{-j})$. Thus, by step 0, $u_i > \widetilde{u}_i \geq \varphi^*_i$.

Let $\bar{v}_j$ be such that $\bar{v}_j > u_j$. By step 0,

$$f_i(\bar{v}_j, u_{-ij}) \leq f_i(u_j, u_{-ij}) = \varphi^*_i \leq \widetilde{u}_i < u_i.$$

Therefore, when the true profile is $(\widetilde{u}_i, \bar{v}_j, u_{-ij})$, agent i can help j by misreporting u_i : Agent i is served in both profiles at price $f_i(\bar{v}_j, u_{-ij})$, however agent j is offered a unit at the cheaper price $f_j(u_{-j})$ when i misreports. This contradicts GSP.

Case A.2. Assume $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u) = 0$, $j \notin S^*$ but $j \in G(\widetilde{u}_i, u_{-i})$.

By SP and MAX, $f_j(\widetilde{u}_i, u_{-ij}) = u_j > f_j(u_{-j})$. So, we are in exactly in the previous case but switching the role of $\widetilde{u}_i$ and u_i. Thus, this case cannot occur.

By repeatedly using step 1.1 to every agent in S^* we have that $G(\widetilde{u}_{S^*}, u_{-S^*}) = S^*$ and $\varphi(\widetilde{u}_{S^*}, u_{-S^*}) = \varphi^*$.

Step 1.2. Let $j \notin S^*$ be such that $\tilde{u}_j < u_j$. Then $G(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = S^*$ and $\varphi(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = \varphi^*$.

Since $\widetilde{u}_j < u_j < f_j(\widetilde{u}_{S^*}, u_{-S^* \cup j})$, then by SP $j \notin G(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j})$. Similarly to step 1.1, by GSP $NU_k(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*})$ for all $k \neq j$.

Assume $G(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) \neq S^*$. Clearly, if $NU_k(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*}) > 0$ for some $k \neq j$, then $k \in S^*$, $k \in G(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j})$ and $\varphi_k(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = \varphi^*_k$.

Thus, there is k such that $NU_k(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*}) = 0$ and either (B.1) $k \in S^*$ but $k \notin G(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j})$; or (B.2) $k \notin S^*$ but $k \in G(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j})$. We show next these cases cannot occur.

Case B.1. $NU_k(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*}) = 0$, $k \in S^*$ and $k \notin G(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j})$.

By SP and MAX,

$$f_k(\widetilde{u}_j, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) > \widetilde{u}_k = f_k(u_j, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}). \quad (3)$$

Let $\bar{v}_k$ be such that $\bar{v}_k > \widetilde{u}_k$. By monotonicity

$$f_j(\bar{v}_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) \leq f_j(\widetilde{u}_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}).$$

First we assume that

$$f_j(\widetilde{u}_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) \geq f_j(\bar{v}_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) > u_j > \widetilde{u}_j.$$

Then, when the true profile is $(\bar{v}_k, \widetilde{u}_j, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j})$, agent j can help agent k by misreporting u_j : Agent j does not get a unit in either profile, however by equation 3 agent k gets a unit at the cheaper price $f_k(u_j, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j})$ when j misreports. This contradicts GSP.

On the other hand, we now assume

$$f_j(\bar{v}_k, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j}) \leq u_j < f_j(\widetilde{u}_k, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j}). \tag{4}$$

Let $\bar{v}_j$ be such that $\bar{v}_j > u_j$. By step 1,

$$f_k(\bar{v}_j, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j}) \leq f_k(u_j, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j}) = \widetilde{u}_k < \bar{v}_k. \tag{5}$$

Thus when true profile is $(\widetilde{u}_k, \bar{v}_j, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j})$, agent k helps j by misreporting $\bar{v}_k$: By equation 5, agent k is served at a price $f_k(\bar{v}_j, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j})$ in either profile; however by equation 4 agent j is served at the cheaper price $f_j(\bar{v}_k, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j})$ when k misreports. This contradicts GSP.

Hence, if $k \in S^*$ then $k \in G(\widetilde{u}_{S^*\cup j}, u_{-S^*\cup j})$ and $\varphi_k(\widetilde{u}_{S^*\cup j}, u_{-S^*\cup j}) = \varphi_k(\widetilde{u}_{S^*}, u_{-S^*})$.

Case B.2. $NU_k(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^*\cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*}) = 0$, $k \notin S^*$ and $k \in G(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^*\cup j})$.

By SP and MAX,

$$f_k(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^*\cup j,k}) = u_k < f_k(u_j, \widetilde{u}_{S^*}, u_{-S^*\cup j,k}).$$

However, this contradicts monotonicity because $\widetilde{u}_j < u_j$.

By repeating step 1.2 to every agent in $N \setminus S^*$, $G(\widetilde{u}) = S^*$ and $\varphi(\widetilde{u}) = \varphi^*$.

Step 2. If $G(u) = G(\widetilde{u})$ then $\varphi(u) = \varphi(\widetilde{u})$.

Proof.

Let $S^* = G(u) = G(\widetilde{u})$, $\bar{v}_S = \max(\widetilde{u}_S, u_S)$ and $\bar{v}_{N\setminus S} = \min(\widetilde{u}_{N\setminus S}, u_{N\setminus S})$ (where max and min are taken coordinate by coordinate).

By step 1, comparing $\bar{v}$ and u, $G(\bar{v}) = S^*$ and $\varphi(\bar{v}) = \varphi(u)$. Similarly, comparing $\bar{v}$ and $\widetilde{u}$, $\varphi(\bar{v}) = \varphi(\widetilde{u})$. Hence $\varphi(u) = \varphi(\widetilde{u})$.

Step 3.

In this final step we prove the theorem by induction on the number of agents. The base of induction is the case $n = 1$. The mechanisms are easy to construct. Given $x \in [0, \infty]$, if $u_1 \geq x$ then $(G, \varphi)(u_1) = (1, x)$. On the other hand, if $u_1 < x$ then $(G, \varphi)(u_1) = (\emptyset, 0)$. These mechanisms are clearly cross-monotonic.

For the induction hypothesis, assume that any GSP and MAX mechanism for k agents, $k < n$, is cross-monotonic. We prove this for the $n-$agent case. Let (G, φ) be a GSP and MAX mechanism defined for the agents $N = \{1, \ldots, n\}$.

Case 1. Assume there is a utility profile u^* such that $G(u^*) = N$.

Let $x^N = \varphi(u^*)$. By step 1, for all $\widetilde{u} \geq x^N$, $G(\widetilde{u}) = N$ and $\varphi(\widetilde{u}) = x^N$.

For every agent $j \in N$, consider the set of utility profiles such that $u_j = 0$, that is

$$U^j = \{u \in \mathbb{R}_+^N \mid u_j = 0\}.$$

By induction, there is a cross-monotonic mechanism (G^j, φ^j) for $N \setminus j$ agents defined on U^j. Thus, $G^j(v) = G(v,0) \cap (N \setminus j)$ and $\varphi^j(v) = \varphi(v,0)_{N\setminus j}$ for all $v \in \mathbb{R}_+^{N\setminus j}$. Let $\tilde{\rho}^j$ be the cross-monotonic set of cost shares that defines this mechanisms.

Let S^*, $S^* \subseteq N \setminus j$, be the maximal coalition that is served by (S^j, φ^j) under any utility profile of $N \setminus j$ agents (by cross-monotonicity this coalition exists).

For every $T \subseteq N \setminus j$ consider the vector of cost shares y^T as follows:

$y^T = x^T$ if $T \subseteq S^*$ and $x^T \in \tilde{\rho}^j$.

$y_i^T = \infty$ if $i \in T \setminus S^*$ and $T \nsubseteq S^*$.

$y_i^T = x_i^{S^*}$ if $i \in S^* \cap T$ and $T \nsubseteq S^*$; where $x^{S^*} \in \tilde{\rho}^j$.

Let $\bar{\rho}^j$ be the set of these cost shares.

Clearly, if $S^* = N \setminus j$, then $\bar{\rho}^j = \tilde{\rho}^j$. If $S^* \neq N \setminus j$, this may not be true, however it generates the same cross-monotonic mechanism (S^j, φ^j) (see below).

First, we show that $\bar{\rho}^j$ is a cross-monotonic set of cost shares. Indeed, consider $L \subset M$ and $k \in L$. If $k \notin S^*$ then $y_k^M = y_k^L = \infty$. Now assume $k \in S^*$. If $L \subset M \subseteq S^*$ then $y_k^M = x_k^M \leq x_k^L = y_k^L$ where $x^M, x^L \in \tilde{\rho}^j$. If $M \nsubseteq S^*$ then $y_k^M = x_k^{S^*} \leq y_k^L$.

Next, we show that $\bar{\rho}^j$ generates the mechanism (S^j, φ^j). Indeed, let v be a utility profile for $N \setminus j$ agents. Then $G^j(v) \subseteq S^*$ by definition of S^*. Clearly, $y^T \in \bar{\rho}^j$ is not reachable at v for any $T \nsubseteq S^*$ because $y_k^T = \infty$ for $k \in T \setminus S^*$. Moreover, $\bar{\rho}^j$ coincides with $\widetilde{\rho}^j$ for any subset in 2^{S^*}, and $G^j(v)$ is the maximal reachable coalition in $\widetilde{\rho}^j$ for the utility profile v. Hence, $G^j(v)$ is the maximal reachable coalition in $\bar{\rho}^j$ for the utility profile v.

Let ρ^j be the embedding of $\bar{\rho}^j$ into U^j by adding a $j-$th coordinate equal to zero.

We define the cost share of coalition T, $T \subsetneq N$ as

$$\tilde{x}^T = \max_{\{x^T \in \rho^j | j \in N \setminus T\}} x^T,$$

where max is taken coordinate by coordinate. The cost share of coalition N is simply x^N. Let ρ^* be the set that contains these cost shares.

We first check that if $G(u) = \bar{S} \neq N$ for some u, then $\varphi(u) = \tilde{x}^{\bar{S}}$. Indeed, by step 1 for any $j \in N \setminus \bar{S}$, $\varphi(u) = \varphi(0, u_{-j}) = x^{\bar{S}}$ where $x^{\bar{S}} \in \rho^j$. Thus for any $i, j \in N \setminus \bar{S}$, $x^{\bar{S}} = \varphi(u) = y^{\bar{S}}$ where $x^{\bar{S}} \in \rho^j$ and $y^{\bar{S}} \in \rho^i$. Furthermore, $\tilde{x}^{\bar{S}} = x^{\bar{S}}$ where $x^{\bar{S}} \in \rho^j$. Hence $\varphi(u) = \tilde{x}^{\bar{S}}$.

We now show ρ^* is a cross-monotonic set of cost shares. Let $S \subset T \subsetneq N$ and $k \in S$. First notice that $x_k^S \geq x_k^T$ holds for any $i \in N \setminus T$, $x^S, x^T \in \rho^i$ by cross-monotonicity on ρ^i. By taking max on both sides of the inequality and maximizing over all agent in $N \setminus T$, $\tilde{x}_k^S \geq \tilde{x}_k^T$ for all $k \in S$.

We now check that $x_i^N \leq \tilde{x}_i^{N\setminus j}$ for all $j \in N$, $i \in N \setminus j$ and $\tilde{x}_i^{N\setminus j} \in \rho^*$. Let S^* be the maximal coalition that is served at (G^j, φ^j) under any u_{-j}. By the choice of $\bar{\rho}^j$, if $i \in N \setminus (j \cup S^*)$, then $\tilde{x}_i^{N\setminus j} = \infty > x_i^N$. On the other hand, if $i \in S^*$, then $\tilde{x}_i^{N\setminus j} = \tilde{x}_i^{S^*}$. To prove the above claim by contradiction, assume there is $i \in S^*$ such that $\tilde{x}_i^{S^*} < x_i^N$. Let $u \in U^j$ such that $\varphi(u) = \tilde{x}^{S^*}$, and $\widetilde{u} = (x_i^N, \max(x_{-i}^N, u_{-i}))$. Since $\widetilde{u} \geq x^N$, then $G(\widetilde{u}) = N$ holds by step 1. Thus $x_i^N = f_i(\widetilde{u}_{-i})$. On the other hand, by step 0, $x_i^N = f_i(\widetilde{u}_{-i}) \leq f_i(u_{-i}) = \varphi_i(u) = \tilde{x}_i^{S^*}$. This is a contradiction.

In particular, cross-monotonicity implies that agent i cannot be served if his utility is smaller than x_i^N. Hence, the mechanism (G, φ) satisfies:

- If $u \geq x^N$ then $G(u) = N$ and $\varphi(u) = x^N$.
- If for some i, $u_i < x_i^N$ then $i \notin G(u)$. Thus, by step 1 $(G, \varphi)(u) = (G, \varphi)(0, u_{-i}) = (G^i(u_{N \setminus i}), \tilde{x}^{G^i(u_{N \setminus i})})$.

Finally, we check (G, φ) is the cross-monotonic mechanism generated by ρ^*. If $u \geq x^N$, then $G(u) = N$ and obviously N is the maximal reachable coalition in ρ^*. Assume u is such that $u_i < x_i^N$ for some agent i. Let $S^* = G(u)$. By cross-monotonicity, no coalition that contains agent i is reachable at u. On the other hand, since (G^i, φ^i) is cross-monotonic, then $S^* = G^i(u_{N \setminus i})$ is the maximal reachable coalition in ρ^i and payments are $x^{S^*} \in \rho^i$. Hence S^* is the maximal reachable coalition in ρ^* because $x^{S^*} = \widetilde{x}^{S^*} \in \rho^*$, and $y^T \geq x^T$ for all $x^T \in \rho^i$ and $y^T \in \rho^*$.

Case 2. Assume there is no u^* such that $G(u^*) = N$.

We will show there is $j \in N$ such that $j \notin G(\widetilde{u})$ for all $\widetilde{u}$. We prove this by contradiction. Assume for any j there is u^j such that $j \in G(u^j)$. Let $\bar{v} = \max(u^1, \ldots, u^n)$ where max is taken coordinate by coordinate. By step 0, at $\bar{v}$ every agent j is offered a unit of good at price not bigger than u_j^j, thus $j \in G(\bar{v})$ for all $j \in N$. This is a contradiction.

Since there is an agent who is not serviced at any profile, say agent j^*, then by step 1 $(G, \varphi)(u) = (G, \varphi)(u_{-j^*}, 0)$ for all u. Hence by induction the mechanism is cross-monotonic.

Proof of Corollary 1.

If the mechanism meets GSP and MIN (MAX), then for every agent i his payment does not decrease (increase) when coalition increases.

Therefore, in order to have a common point at every coalition, it must be that $x_i^N = x_i^i$ for all i. Hence, the cost share of agent i is fixed.

Proof of Proposition 1.

Recall that $1_N = (1, \ldots, 1) \in \mathbb{R}_+^N$. For a non-negative number x, let $x \cdot 1_N = (x, \ldots, x) \in \mathbb{R}_+^N$.

The proof of Proposition 1 is divided in cases 1 and 2.

By ETE, $G(x \cdot 1_N) = N$ or $G(x \cdot 1_N) = \emptyset$ for all $x \geq 0$, since all the agents should either be served or not served at a symmetric utility profile.

Case 1. Assume $G(x \cdot 1_N) = \emptyset$ for all $x > 0$, then $G(u) = \emptyset$ for all $u \in \mathbb{R}_+^N$.

Proof.

Step 1.1. If $NU_k(u) = 0$ for all $u \in \mathbb{R}_+^N$ and $k \in N$, then $G(u) = \emptyset$ for all $u \in \mathbb{R}_+^N$.

Proof. If $NU_k(u) = 0$ but $G(u) = S \neq \emptyset$ for some utility profile u then $\varphi_i(u) = u_i$ for all $i \in S$. Thus by SP, for $k \in S$ and $v_k > u_k : k \in G(v_k, u_{-k})$ and $\varphi_k(v_k, u_{-k}) = u_k$, thus $NU_k(v_k, u_{-k}) > 0$. This is a contradiction.

Step 1.2. Assume $G(x \cdot 1_N) = \emptyset$ for all $x > 0$, then $NU(u) = 0$ for all $u \in \mathbb{R}^N_+$.

Proof. Assume there is an agent k such that $NU_k(u) > 0$ at some utility profile u. Let $u^{max} = \max(u_1, \ldots, u_n) \cdot 1_N$. Then, $G(u^{max}) = \emptyset$. Thus, when the true profile is u^{max}, agents in N help k by misreporting u : Agent k is strictly better off because he is getting a unit at a price below u_k, while any other agent j may or may not get a unit at a price less than or equal to u_j. This contradicts GSP.

Steps 1.1 and 1.2 prove case 1.

Case 2. If there exists $x^* \geq 0$ such that $G(x^* \cdot 1_N) = N$, then (G, φ) is welfare equivalent to a cross-monotonic mechanism that satisfy equal-sharing.

Proof.

By ETE, there exists $y^* \geq 0$ such that $\varphi_i(x^* \cdot 1_N) = y^*$ for all $i \in N$. The rest of the proof of this case is divided in steps 2.1, 3 and 4.

Step 2.1. For all $u > y^* \cdot 1_N$, $G(u) = N$ and $\varphi(u) = y^* \cdot 1_N$.

Proof.

First assume that $x^* > y^*$. Let $v = x^* \cdot 1_N$. By SP, $1 \in G(v_{-1}, u_1)$ and $\varphi_1(v_{-1}, u_1) = y^*$. Thus, by GSP, $G(v_{-1}, u_1) = N$ and $\varphi_i(v_{-1}, u_1) = y^*$ for all i. Changing the profiles one agent at a time $G(u) = N$ and $\varphi_i(u) = y^*$ for all $i \in N$

Now, assume that $y^* = x^*$. Consider $\widetilde{x}$ such that $\widetilde{x} > x^*$. Let $\widetilde{v} = \widetilde{x} \cdot 1_N$. By ETE, $G(\widetilde{v}) = \emptyset$ or $G(\widetilde{v}) = N$. If $G(\widetilde{v}) = \emptyset$ then when the true profile is $\widetilde{v}$ coalition N can improve by misreporting $x^* \cdot 1_N$, since all agents are served at price $y^* = x^*$ at that profile. On the other hand, if $G(\widetilde{v}) = N$, then $\varphi(\widetilde{v}) = y^* \cdot 1_N$. Indeed, $\varphi(\widetilde{v}) \geq y^* \cdot 1_N$ because $\widetilde{x} > x^* = y^*$. If $\varphi(\widetilde{v}) > y^* \cdot 1_N$ then when the true profile is $\widetilde{v}$ all agents can improve by misreporting $x^* \cdot 1_N$, since all agents are served at price $y^* = x^*$ at that profile.

Therefore $G(\widetilde{x} \cdot 1_N) = N$, $\varphi(\widetilde{x} \cdot 1_N) = y^* \cdot 1_N$ and $\widetilde{x} > y^*$. By the initial case, $G(u) = N$ and $\varphi(u) = y^* \cdot 1_N$.

We finish the proof of case 2 by induction in the number of agents. We assume that any ETE mechanism for less than n agents is welfare equivalent to a cross-monotonic mechanism that satisfy equal sharing. We will prove it for a mechanism for n agents. We will divide the proof in steps 3 and 4 (and several cases in between).

Step 3. If an agent is served, he will not pay more than y^* at any utility profile. That is, if $i \in G(u^*)$ for some $u^* \in \mathbb{R}^*_+$ then $\varphi_i(u^*) \geq y^*$.

Proof.

We will prove this step by analyzing cases 3.1 and 3.2.

Case 3.1. $G(u^*) \neq N$.

In order to derive a contradiction, we assume that $\varphi_i(u^*) < y^*$ for some agent i.

Without loss of generality, also assume that $j \notin G(u^*)$ and $\varphi_i(u^*) < u^*_i$, so agent i gets a positive net utility at u^*.[14]

Consider the profile $\widetilde{u} = (0, u^*_{-j})$. Then by GSP, $i \in G(\widetilde{u})$ and $\varphi_i(\widetilde{u}) = \varphi_i(u^*)$. Otherwise, j would help i by reporting the profile that give i higher utility.

Let $U^j = \{u \in \mathbb{R}^N | u_j = 0\}$ be the set of utility profiles where agent j has utility zero. By induction, the restriction of the mechanism to U^j is welfare equivalent to a cross-monotonic mechanism for $N \setminus j$ agents that satisfies equal sharing.

Since $\widetilde{u} \in U^j$ and $i \in G(\widetilde{u})$ and the mechanism restricted to U^j is cross-monotonic with equal sharing, then we can find a utility profile $w \in U^j$ such that $w \geq \widetilde{u}$ and $G(w) = N \setminus j$ and $\varphi_i(w) \leq \varphi_i(\widetilde{u}) = \varphi_i(u^*) < y^*$.

Let $x^{N\setminus j} = \varphi(w)$. Clearly $x^{N\setminus j}_k = x^{N\setminus j}_i < y^*$ for all $k \neq j$.

Let $\epsilon > 0$ be such that $y^* - \epsilon > x^{N\setminus j}_i$ and consider $u = ((y^* + \epsilon) \cdot 1_N)$. Then by step 2.1, $G(u) = N$ and $\varphi(u) = x^N$. By SP, $j \notin G(y^* - \epsilon, u_{-j})$. Thus by GSP $G(y^* - \epsilon, u_{-j}) = N \setminus j$ and $\varphi(y^* - \epsilon, u_{-j}) = x^{N\setminus j}$.

Since $u_k > x^{N\setminus j}_k$ for all $k \in N\setminus j$, then by GSP $G((y^* - \epsilon) \cdot 1_N) = N\setminus j$ and $\varphi((y^* - \epsilon) \cdot 1_N) = x^{N\setminus j}$. This contradicts ETE.

Case 3.2. $G(u^*) = N$

In order to derive a contradiction, we assume that $\varphi_i(u^*) < y^*$ for some agent i.

If agent $i \in G(u^*)$ is such that $\varphi_j(u^*) = u^*_j$, then at the profile $(0, u^*_{-j})$ agent j is not served, that is $j \notin G(0, u^*_{-j})$. Thus we can apply the case 3.1.

On the other hand, if $\varphi_k(u^*) < u^*_k$ for all $k \in N$, then consider the utility profile $v^* = \max\{u^*_1, u^*_2, \ldots, u^*_n\} \cdot 1_N$. By GSP (replacing one agent at a time): $G(v^*) = N$ and $\varphi(v^*) = \varphi(u^*) < y^*$. This contradicts step 2.1 because $y^* \cdot 1_N > v^*$ and thus $\varphi(y^* \cdot 1_N) = \varphi(v^*)$.

Step 4. The mechanism is a cross-monotonic mechanism with equal-sharing.

Proof.

By the induction hypothesis, the mechanism restricted to U^j is a cross-monotonic mechanism with equal-sharing. The cost-shares of the mechanism are given by: $x^N = y^* \cdot 1_N$ and for $S \subseteq N \setminus j$ for some j, x^S is the cost share of coalition S at U^j (which exist because the mechanism restricted to U^j is cross-monotonic). These cost-shares are cross-monotonic because they are cross-monotonic at every U^j, and $x^N_i = y^* \leq x^S_i$ for every $i \in S \subset N \setminus j$ by step 3.

Consider a utility profile u.

Case 4.1. If $u_i \geq y^*$ for all $i \in N$, then the mechanism is welfare equivalent to the mechanism such that $G(u) = N$ and $\varphi_i(u) = y^*$.

If $u_i > y^*$ for all $i \in N$, then by step 2.1, $G(u) = N$ and $\varphi_i(u) = y^*$ for all $i \in N$.

If $u_i > y^*$ for all $i \in S$ and $u_j = y^*$ for all $j \in N \setminus S$, then by step 3 no agent will pay less than y^*. If an agent $k \in S$ is paying more than y^* at u, then coalition N can help k by misreporting $x^* \cdot 1_N$, since all agents pay exactly y^* at that profile.

[14] If $\varphi_i(u^*) = u^*_i$, then just increase the utility of agent i by $\epsilon > 0$ and by SP he will get a unit of good at a cheaper price than his utility.

Case 4.2. If $u_i < y^*$ for some i, then by step 3 $i \notin G(u)$. By SP, $i \notin G(0, u_{-i})$. Moreover, by GSP $NU_j(0, u_{-i}) = NU_j(u)$ for all $j \neq i$. To see this, if $NU_j(0, u_{-i}) > NU_j(u)$ then when the true profile is u, agent i helps j by misreporting 0. Similarly, if $NU_j(0, u_{-i}) < NU_j(u)$, then when the true profile is $(0, u_{-i})$, agent i helps j by misreporting u_i. Therefore, $NU_j(0, u_{-i}) = NU_j(u)$. Since the restriction to U^j is welfare equivalent to a cross-monotonic mechanism with cost-shares not smaller than y^* and x^N is not reachable because $u_i < y^*$, then $G(u)$ is the maximum reachable coalition u.

Proof of Proposition 2.

First notice if agent i is not served at any profile, then by GSP $NU_k(u) = NU_k(\widetilde{u}_i, u_{-i})$ for all $k \neq i$, u and $\widetilde{u}_i$. Thus we can remove this agent from the mechanism without any welfare consequence.

We prove the proposition by contradiction. Assume without loss of generality that every agent in N is served in at least one profile and that there is no agent who has priority. Then for every agent i there exist profiles u^i and $\widetilde{u}^i$ such that $i \in G(u^i)$, $i \notin G(\widetilde{u}^i)$, $u^i_i, \widetilde{u}^i_i > \bar{x}_i$ where $\bar{x}_i = \varphi_i(u^i)$.

Let $\bar{v} >> \max_{k \in N}(u^k, \widetilde{u}^k)$ where max is taken coordinate by coordinate over all utility profiles u^k, $\widetilde{u}^k$.

By GSP, $G(\bar{v}) \neq \emptyset$, otherwise coalition N misreport u^1 when true profile is $\bar{v}$. Assume $G(\bar{v}) = i^*$. By GSP, $\varphi_{i^*}(\bar{v}) = \bar{x}_i$, otherwise coalition N misreport u^1 when true profile is $\bar{v}$ or vice-versa.

By SP, $k \notin G(\widetilde{u}^{i^*}_k, \bar{v}_{-k})$ for all $k \neq i^*$. Thus by GSP, $G(\widetilde{u}^{i^*}_k, \bar{v}_{-k}) = i^*$ and $\varphi_{i^*}(\widetilde{u}^{i^*}_k, \bar{v}_{-k}) = \bar{x}_{i^*}$. Changing the profiles one agent at a time, $G(\widetilde{u}^{i^*}_{-i^*}, \bar{v}_{i^*}) = i^*$ and $\varphi_{i^*}(\widetilde{u}^{i^*}_{-i^*}, \bar{v}_{i^*}) = \bar{x}_{i^*}$.

Since $\widetilde{u}_{i^*} > \bar{x}_{i^*}$ then by SP $G(\widetilde{u}^{i^*}) = i^*$. This is a contradiction.

References

[1] Deb, R., Razzolini, L., 1999. Voluntary cost sharing for an excludable public project. Mathematical Social Sciences 37, 123-138.

[2] Devanur, N., Mihail, M., Vazirani, 2005. Strategyproof cost-sharing mechanisms for set cover and facility location games. Decision Support Systems, 11-22.

[3] Dutta, B, Ray, D., 1989. A concept of egalitarianism under participation constraints. Econometrica, 57, 615-635.

[4] Ehlers, L., 2002. Coalitional Strategy-Proof House Allocation. J. Econ. Theory 105, 298-317.

[5] Ehlers, L, Klaus, B., 2003. Coalitional strategy-proof and resource-monotonic solutions for multiple assignment problems. Soc. Choice Welfare 21, 265-280.

[6] Goldberg, A., Hartline, J., 2004. Collusion-Resistant Mechanisms for Single-Parameter Agents. Mimeo Microsoft.

[7] Immorlica, N., Mahdian, M., Mirrokni, V., 2005. Limitations of cross-monotonic cost sharing schemes. Mimeo MIT.

[8] Juarez, R., 2006. The worst absolute surplus loss in the problem of commons: Random Priority vs. Average Cost. Forthcoming Econ. Theory.

[9] Juarez, R., 2007a. Group strategyproof cost sharing: budget balance vs. efficiency. Mimeo Rice University.

[10] Juarez, R., 2007b. Group strategyproof cost sharing: Mechanisms without indifferences. Mimeo Rice University.

[11] Moulin, H., 1999. Incremental Cost Sharing: characterization by coalitional strategy-proofness. Soc. Choice Welfare 16, 279-320.

[12] Moulin, H. 2007. The price of anarchy of serial cost sharing and other methods. Forthcoming Econ. Theory.

[13] Moulin, H., Shenker, S., 1992. Serial Cost Sharing. Econometrica 50, 1009-1039.

[14] Moulin, H., Shenker, S., 1994. Average Cost Pricing Versus Serial Cost Sharing: an axiomatic comparison. J. Econ. Theory 64, 178-201.

[15] Moulin, H., Shenker, S., 2001. Strategyproof sharing of submodular costs: budget balance versus efficiency. Econ. Theory 18, 511-533.

[16] Mutuswami, S., 2005. Strategyproofness, Non-Bossiness and Group Strategyproofness in a cost sharing model. Econ. Letters 89, 83-88.

[17] Norde, H., Reijnierse, H., 2002. A dual description of the class of games with a cross monotonic scheme. Games Econ. Behav. 41, 322-343.

[18] P'al, M., Tardos, E., 2003. Group strategyproof mechanisms via primal-dual algorithms. In Proceedings of 44th Annual IEEE Symposium on Foundations of Computer Science (FOCS), pp. 584 - 593.

[19] Papai, S., 2000. Strategyproof Assignment by Hierarchical Exchange. Econometrica 68, 1403-1433.

[20] Papai, S., 2001. Strategy-proof and nonbossy assignments. J. Public Econ. Theory 3, 257-271.

[21] Roughgarden, T., Sundararajan, M., 2006a. Approximately efficient cost-sharing mechanisms. Mimeo Stanford.

[22] Roughgarden, T., Sundararajan, M., 2006b. New TradeOffs in Cost Sharing Mechanisms. STOC.

[23] Roughgarden, T., Mehta, A., Sundararajan, M., 2007. Beyond Moulin Mechanisms. ACM Conference on Electronic Commerce.

[24] Sprumont, Y., 1990. Population Monotonic Allocation Schemes for Cooperative Games with Transferable Utility. Games Econ. Behav. 2, 378-394.

[25] Svensson, L., Larsson, B., 2002. Strategy-proof and nonbossy allocation of indivisible goods and money. Econ. Theory 20(3), 483-502.