Unequal Numbers of observations and Partial Efficiency Gain: A Note

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Efficiency gain from Aitken estimation in the seemingly unrelated regressions [Zellner (1962)] is not obtained when regressors are identical [Dwivedi and Srivastava (1978)]. Swamy and Mehta (1975) show that there is efficiency gain from Aitken estimation when there are unequal numbers of observations, and a Monte Carlo study by Schmidt (1975) shows small sample evidence. Most recently, Farebrother (1992) deals with the model in the context of no contemporaneous observations. However, there has not been discussed estimation efficiency in the context of unequal numbers of observations and identical regressors. This note is to fill the gap by showing that the efficiency gain is only partial.
1. Introduction

Consider, without losing generality, a set of two seemingly unrelated regressions

\[ Y_1 = X_1 \beta_1 + \varepsilon_1; \quad Y_2 = X_2 \beta_2 + \varepsilon_2 \quad (1) \]

with the standard assumptions for the model.

If we add an assumption of unequal observations, \( T \) observations for the first equation and \( T+m \) observations, \( m \geq 1 \), for the second, we can partition \( X_2 \) and \( Y_2 \):

\[ X_2 = \begin{bmatrix} X_2^* \\ X_2^o \end{bmatrix}; \quad Y_2 = \begin{bmatrix} Y_1^* \\ Y_2^o \end{bmatrix} \quad (2) \]

with \( X_2^* \) and \( Y_2^* \) containing the first \( T \) observations in \( X_2 \) and \( Y_2 \), and \( Y_2^o \) and \( X_2^o \) for the \( m \) additional observations.

Denoting by \( \hat{\beta} = [\hat{\beta}_1^* \hat{\beta}_2^o]' \) Aitken estimator of \( \beta = [\beta_1^* \beta_2^o]' \) in

\[ \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \]

\[ \begin{bmatrix} \sigma^{11}X_1'X_1 & \sigma^{12}X_1'X_2^* \\ \sigma^{21}X_2^o'X_1 & \sigma^{22}X_2^o'X_2^* + \frac{1}{\sigma_{22}}X_2^o'X_2^o \end{bmatrix}^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \quad (3) \]

where \( \sigma_{ij} \) and \( \sigma^{ij} \) are elements of \( \Sigma = \text{Cov}(\varepsilon_1, \varepsilon_2) \) and \( \Sigma^{-1} \), respectively.

From (3) readily follows:

\[ \text{Cov}(\hat{\beta}_2) = B_{22} = \left[ \sigma^{22}X_2^o'X_2^o + \frac{1}{\sigma_{22}}X_2^o'X_2^o - \sigma^{21}X_2^o'X_1(\sigma^{11}X_1'X_1)^{-1}\sigma^{12}X_1'X_2^o \right]^{-1} \quad (4) \]
\[
\text{Cov}(\hat{\beta}_1) = B_{11} = (\sigma^{11}X'_1X_1)^{-1} + (\sigma^{11}X'_1X_1)^{-1}(\sigma^{21}X'_2X_2^*)B_{22}(\sigma^{12}X'_1X_*^*) (\sigma^{11}X'_1X_1)^{-1} .
\]

(5)

2. Derivation and Remarks.

In the context of (2), we can show that if regressors are identical for \( y_1 \) and \( y_2 \) efficiency gain from Aitken estimation is limited to \( \beta_1 \) only:

\[
\text{Cov}(\hat{\beta}_1) < \text{Cov}(b_1) ;
\]

\[
\text{Cov}(\hat{\beta}_2) = \text{Cov}(b_2)
\]

(6)

where \( b_1 \) and \( b_2 \) are separate OLS estimators of \( \beta_1 \) and \( \beta_2 \), respectively.

If regressors are identical for both \( y_1 \) and \( y_2 \), for \( X_2^* \) in (2)

\[
X_2^* = X_1^*.
\]

(7)

Substituting (7) into (4) and (5), and noting that

\[
\sigma_{ij} = \frac{\sigma_{ii}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{12}} ; \quad \sigma_{i'j'} = \frac{\sigma_{i'j'}}{\sigma_{11}\sigma_{22} - \sigma_{21}\sigma_{12}},
\]

\[
\text{Cov}(\hat{\beta}_2) = B_{22}
\]

\[
= \left[ \sigma^{22}X'_1X_1 + \frac{1}{\sigma_{22}}X'_2X_2^* - \frac{\sigma^{12}\sigma_{12}^{12}}{\sigma^{11}}X'_1X_1 \right]^{-1}
\]

\[
= \left[ \frac{\sigma^{11}\sigma_{22} - \sigma^{12}\sigma_{12}^{12}}{\sigma^{11}}X'_1X_1 + \frac{1}{\sigma_{22}}X'_2X_2^* \right]^{-1}
\]

\[
= \left[ \frac{1}{\sigma_{22}}X'_1X_1 + \frac{1}{\sigma_{22}}X'_2X_2^* \right]^{-1}
\]
\[
= \sigma_{22} \left( x_1' x_1 + x_2' x_2 \right)^{-1} = \text{Cov}(b_2) \tag{8}
\]

\[
\text{Cov}(\hat{\beta}_1) = B_{11}
\]

\[
= \left( \sigma_1^2 x_1' x_1 \right)^{-1} + \sigma_{22} \frac{\sigma_{21} \sigma_{12}}{(\sigma_{11})^2} \left( x_1' x_1 + x_2' x_2 \right)^{-1}
\]

\[
< \left( \sigma_1^2 x_1' x_1 \right)^{-1} + \sigma_{22} \frac{\sigma_{21} \sigma_{12}}{(\sigma_{11})^2} \left( x_1' x_1 \right)^{-1}
\]

\[
= \frac{1}{\sigma_1^2} + \sigma_{22} \frac{\sigma_{21} \sigma_{12}}{(\sigma_{11})^2} \left( x_1' x_1 \right)^{-1}
\]

\[
= \left[ \frac{\sigma_{11} \sigma_{22} - \sigma_{21} \sigma_{12}}{\sigma_{22}} + \sigma_{22} \frac{\sigma_{21} \sigma_{12}}{(\sigma_{11})^2} \right] \left( x_1' x_1 \right)^{-1}
\]

\[
= \sigma_{11} \left( x_1' x_1 \right)^{-1} = \text{Cov}(b_1) \tag{9}
\]

which completes the proof of (6).

For seemingly unrelated regressions in the context herein addressed, one may have an intuitive inclination to no efficiency gain from Aitken estimation in light of Dwivedi and Srivastrava (1978), or to efficiency gain in light of Swamy and Mehta (1975). These potentially inaccurate intuitive inclinations with regard to the efficiency gain are cleared in this note.
REFERENCES


