Probabilistic Assignment of Objects: Characterizing the Serial Rule

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Abstract

We study the problem of assigning a set of objects (i.e., indivisible goods) to a set of agents, when each agent is supposed to receive only one object and has strict preferences over the objects. In the absence of monetary transfers, we focus on the probabilistic rules, which only take the ordinal preferences as input (the ordering over objects by each agent is submitted, but the relative cardinal intensities of their preferences are not).

We present a characterization of the serial rule, proposed by Bogomolnaia and Moulin (2001) in this model. The serial rule is the only rule satisfying sd efficiency, sd no-envy, and bounded invariance (where "sd" stands for "stochastic dominance"). A special representation of feasible assignment matrices by means of consumption processes is the key to the simple and intuitive proof of our main result. This technique also allows us to present a simple unifying argument for a number of related earlier and concurrent results.

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Keywords: probabilistic assignment; Serial rule; sd efficiency; sd no-envy; bounded invariance; consumption process

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1 Introduction

We study the problem of assigning a set of "objects" (i.e., indivisible goods) to a set of agents, when each agent is to receive only one object and has strict preferences over the objects. In the absence of monetary transfers, randomization is the method of choice to guarantee fairness. An extensive recent literature, starting from Bogomolnaia and Moulin (2001), is devoted to the study of probabilistic assignment rules in this setting. Earlier work (see Hylland and Zeckhauser (1979), Zhou (1990)) assumed that agents announce their cardinal preferences. The recent literature considers ordinal rules, which only take agents' rankings over the objects as input.

The classical random priority rule orders agents using a uniform lottery, and lets them pick their most preferred objects in that order. An alternative is the "serial rule (S)," introduced by Bogomolnaia and Moulin (2001). The rule is described by means of a "simultaneous eating" algorithm. Agents acquire probabilities of objects continuously over the unit interval of time $[0, 1]$, simultaneously and at the same unit rate. Given a preference profile R, each agent starts with his most preferred object. When the object he consumes is exhausted, he switches to his next most preferred object among the ones that are still available. At each $\tau \in [0, 1]$, $S(R)[\tau]$ is a partial serial assignment that agents have acquired by τ. The final assignment $S(R) = S(R)[1]$ is given by vectors of probabilities agents have consumed.

The restriction to ordinal input calls for first order stochastic dominance as a way to compare assignments. Given agent i's ordinal preferences, one assignment (vector of probabilities receiving different objects) weakly dominates another for agent i, if and only if, for each object a, the total probability of his receiving objects that he prefers to a is at least as large under the first assignment as under the second.[1] In this spirit, Bogomolnaia and Moulin (2001) define two natural requirements on a probabilistic assignment. "Sd efficiency"[2] requires that no other assignment weakly dominates the given one for all agents. "Sd no-envy" requires for each agent's assignment to dominate, in his opinion, the assignment of any other agent.

The serial rule fares better than the random priority rule in that it satisfies sd

[1]This is equivalent to saying that an agent with *any* vNM index, consistent with announced ordinal ordering, finds the first assignment at least as desirable as the second (i.e. has at least as large expected utility at the first one as at the second). Thus, if the first assignment weakly dominates the second, the agent always prefers the first, or at least feels indifferent between them. If neither dominates another, then agents with *some* vNM indices would prefer the first to the second, while agents with *some other* vNM indices would prefer the second.

[2]For short, we use the prefix "sd" for stochastic dominance in other expressions below. In Bogomolnaia and Moulin (2001), this requirement is referred to as "ordinal efficiency." In this paper, we adopt the terminology and the notation of Thomson (2010).

efficiency and sd no-envy, neither one of which is satisfied by the random priority rule, though those properties do not pin it down (see Remark in Section 3).[3]

Much work has been done over the last decade on probabilistic rules that only take ordinal preferences as input. The serial rule occupies a central place in this literature. However, until recently, its axiomatic characterization had been elusive. This paper proposes such a characterization, by means of sd efficiency, sd no-envy, and an axiom, which we call "bounded invariance". This last requires that, whenever the preferences of one agent change so that the ranking over his upper counter set of an object, say a, remains the same, the probability share of a assigned to *each* agent remains the same. We show that this characterization holds even under weaker requirements. Moreover, we introduce a new, simple, and intuitive proof technique, which allows us to easily obtain an array of related characterizations.

The key to our result is a special representation of assignment matrices by means of "consumption processes". This representation is introduced by Heo (2010) as "preference-decreasing consumption schedules": each assignment matrix P is represented as the output of a process along which agents continuously acquire probabilities of objects over the unit interval of time, $[0, 1]$. It mimics the simultaneous eating algorithm: each agent consumes probabilities of objects at unit speed in decreasing order of his preferences. However, each agent may switch from one object to another even when the former object is not yet exhausted. Agent i ends consuming an object, say a, exactly when she acquires the probability p_{ia}, given by P_i. We represent this process by $(P[\tau])_{\tau\in[0,1]}$. Given a rule satisfying the properties that we impose, we compare the simultaneous eating algorithm with the consumption processes representing assignment matrices selected by the rule. By imposing our properties, we obtain that the assignment matrix selected by the rule should coincide with the assignment matrix selected by the serial rule.

Three recent papers are closely related to the current manuscript. Heo (2010) provides a characterization of the serial rule by means of sd efficiency, "sd equal-division lower bound," "limited invariance," and "consistency," an axiom pertaining to variable populations.[4] Independently, Kesten et al. (2010) were the first to characterize the rule by means of sd efficiency, sd no-envy, and one additional invariance axiom, which they call "upper invariance".[5] Independently and con-

[3]On some restricted preference domains, however, the two properties characterize the serial rule. See Bogomolnaia and Moulin (2002) and Liu and Pycia (2011).

[4]This paper characterizes a generalized version of the serial rule when each agent receives possibly more than one object; our setting is a special case in which each agent receives only one object. We discuss this generalization in Section 4.

[5]Our paper combines independent efforts of current authors. Heo formulated a weaker invariance axiom than upper invariance and derived the result that is the main result of the current

currently, Hashimoto and Hirata (2011) formulated another invariance axiom for a different domain on which the "null object"[6] exists. Applying our result directly to the variants of the model in which several copies of objects may exist, or agents could receive the null object, we obtain these characterizations as corollaries, given that our invariance axiom implies theirs. The studies by Kesten et al. (2010) and Hashimoto and Hirata (2011) contain two additional characterizations that do not invoke invariance axioms and are distinct from ours (Theorem 1 in Kesten et al. (2010) and Theorem 3 in Hashimoto and Hirata (2011)). After we became aware of these results, we found that our proof technique also allows us to obtain straightforward proofs of these results.[7]

The paper is organized as follows. We describe the model, notation, and axioms in Section 2. We present our main result in Section 3. We show how all the related results, mentioned above, can be unified using our proof technique in Section 4 (we relegate a short argument, needed for one case, to the appendix).

2 Model

Let $A = \{o_1, \ldots, o_n\}$ be a set of objects and $N = \{1, 2, \ldots, n\}$ a set of agents.[8] Each $i \in N$ has a strict preference R_i over A. Let $\mathcal{R}$ be the set of all such preferences. Let $R = (R_i)_{i \in N} \in \mathcal{R}^N$ be the preference profile. For each $i \in N$ and each $R_i \in \mathcal{R}$ and each $a \in A$, denote by $U^0(R_i, a) = \{b \in A : b \; R_i \; a\}$ the strict upper contour set of R_i at a, and by $U(R_i, a) = U^0(R_i, a) \cup \{a\}$ the (weak) upper contour set of R_i at a. For each $S \subseteq A$, let $R_i|_S$ be the preference R_i restricted to S. For each $a \in A$, let $R_i(a) \equiv R_i|_{U(R_i,a)}$, that is, the preference R_i restricted to $U(R_i, a)$.

A **probabilistic assignment matrix** is an $|N| \times |A|$ matrix $P = (p_{ia})_{i \in N, a \in A}$ where p_{ia} is the probability that agent i receives object a. Let P_i be the row of the probabilities of agent i receiving the various objects and let P^a be the column of the probabilities of object a being assigned to the various agents. We refer to

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paper, exploiting a proof technique initiated in her earlier work (Heo, 2010). Bogomolnaia independently improved on a result in Kesten et al. (2010) by invoking the same axioms as Heo, and developing a simpler proof.

[6]The null object is defined as receiving no object.

[7]We recently became aware that, as a latest development of their line of work, Hashimoto and Hirata (2011) and Kesten et al. (2010) are merging their efforts, proving more general characterization (Hashimoto et al., 2012). They impose limited invariance of Heo (2010) in place of upper invariance.

[8]In the main body of the paper, we assume that $|A| = |N|$ and all objects are different and acceptable. One can think of the null object as being there but being always ranked last. We extend our results to several generalized environments in Section 4. However, we concentrate on the simplest model both to sharpen the clarity of exposition and to obtain a characterization on the smallest domain.

the vector P_i as "agent i's assignment" at P.

A probabilistic assignment matrix P is **feasible** if and only if it is bistochastic, i.e. (i) for each $i \in N$ and each $a \in A$, $p_{ia} \in [0,1]$, (ii) $\sum_{i \in N} p_{ia} = 1$, and (iii) $\sum_{a \in A} p_{ia} = 1$. By the Birkhoff-von Neumann theorem (Birkhoff (1946), von Neumann (1953)), each bistochastic matrix can be represented as a convex combination of permutation matrices,[9] i.e., as a probability distribution over deterministic assignments. Let $\mathcal{P}$ be the set of all feasible assignment matrices. A **rule** is a mapping from $\mathcal{R}^N$ to $\mathcal{P}$. Denote a generic rule by φ.

2.1 Axioms

The rules that we study only take rankings over objects as input. We compare an agent's assignments by means of first order stochastic dominance. Let R_i be agent i's preference and P_i, Q_i be two assignments for agent i. We say that $\boldsymbol{P_i}$ **weakly stochastically dominates** $\boldsymbol{Q_i}$ **at** $\boldsymbol{R_i}$, which we write as $P_i \; R_i^{sd} \; Q_i$, if for each $a \in A$, $\sum_{b \in U(R_i,a)} p_{ib} \geq \sum_{b \in U(R_i,a)} q_{ib}$.

The following are two important properties of assignment matrices. Let R be a preference profile and P an assignment matrix. We say that P is **sd efficient at** $\boldsymbol{R}$ if there is no other feasible assignment matrix Q such that for each $i \in N$, $Q_i \; R_i^{sd} \; P_i$ and $P \neq Q$. We say that P is **sd envy-free at** $\boldsymbol{R}$ if for each pair $i, j \in N$, $P_i \; R_i^{sd} \; P_j$. Our first two axioms are that for each preference profile, the assignment matrix determined by a rule should satisfy these efficiency and fairness properties. They are standard in our model. The serial rule satisfies both of them (Bogomolnaia and Moulin, 2001).

Sd efficiency: For each $R \in \mathcal{R}^N$, $\varphi(R)$ is *sd efficient* at R.

Sd no-envy: For each $R \in \mathcal{R}^N$, $\varphi(R)$ is *sd envy-free* at R.

We now introduce a third axiom, which is new. It is an invariance requirement. It restricts how a rule reacts to changes in the preferences of a single agent. Suppose that the preference of an agent changes but his ranking above a certain object, say a, remains the same. We require that this change should not affect the probability share of a assigned to *each* agent.[10]

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[9] P is a permutation matrix if $P \in \mathcal{P}$ and for each agent i and each object a, $p_{ia} \in \{0,1\}$.

[10] A requirement in the spirit of *bounded invariance* has also been formulated in the probabilistic voting model (Gibbard, 1977). When each assignment matrix is viewed as a collective decision, the requirement can be rephrased as follows. Given other agents' preferences, suppose that an agent's preference changes but his upper contour set at an object, say a, remains the same. Then, the total probability assigned to objects in this upper contour set should remain the same for *each agent*. It is obvious that this requirement implies our *bounded invariance*.

Bounded invariance: For each $R \in \mathcal{R}^N$, each $i \in N$, each $a \in A$, and each $R'_i \in \mathcal{R}$, if $R_i(a) = R'_i(a)$, then for each $j \in N$, $\varphi_{ja}(R) = \varphi_{ja}(R'_i, R_{-i})$.[11]

2.2 Consumption Process

Next, we define an alternative representation of an assignment matrix by means of "consumption process". This idea is introduced by Heo (2010) under the name of "preference decreasing consumption schedule". It is the key to the simple proof of our main result.[12]

Let $R \in \mathcal{R}^N$ and $P \in \mathcal{P}$. We interpret the assignment matrix P as the output of a specific consumption process over the time interval $[0, 1]$. Each agent, say i, consumes probabilities at unit speed in decreasing order of his preferences. He starts by consuming his most preferred object, say a, at time 0, and switches to his second most preferred object, say b, at time p_{ia}. Then, he switches to his third most preferred object at time $p_{ia} + p_{ib}$, and so on. The time interval during which he consumes each object o is exactly the probability p_{io}, given by P_i.

Let $R \in \mathcal{R}^N$. At each time $\tau \in [0, 1]$, we define $P[\tau]$ as an $|N| \times |A|$ matrix representing the partial assignment of P that is "acquired" by time τ, given the consumption process described above. In particular, $P[0]$ is a zero matrix, while $P[1] = P$. The consumption process representing P is denoted by $(P[\tau])_{\tau \in [0,1]}$. Let $P_i[\tau] = (P[\tau])_i$ be what has been acquired by agent i at time τ in the consumption process representing P.

For each $R \in \mathcal{R}^N$, the assignment matrix selected by the serial rule, $S(R)$, is obtained by a similar algorithm. There is no assignment matrix that specifies the consumption process to begin with. Instead, each agent consumes probabilities of objects in decreasing order of his preferences. He starts from his most preferred object and consumes it as long as it is available. Whenever an object he consumes is exhausted, he switches to his best object among the ones that are still available.[13] For each $\tau \in [0, 1]$, we denote by $S(R)[\tau]$ the corresponding partial serial assignment by time τ. The defining property of consumption process representing $S(R)$ is that at each time, each agent consumes his most preferred object among the ones that are still available.

Note that the consumption process is continuous in τ and piece-wise linear: it has only a finite number of "switches" (at most $n(n-1)$), when some agent changes from one object to another. The following two lemmas are immediate.

Lemma 1 The total probability of agent i receiving objects at least as desirable as object a, is the time at which he ends consuming a in the consumption process.

[11]It is easy to see that this property is equivalent to a seemingly stronger one, demanding that for each $o \in U(R_i, a)$ and each $j \in N$, $\varphi_{jo}(R) = \varphi_{jo}(R'_i, R_{-i})$.

[12]See Heo (2010) for an extensive discussion and related notion.

[13]Note that in our algorithm several agents can consume an object simultaneously.

Lemma 2 Let R be a preference profile and P and Q be two assignment matrices. Suppose that, for some $t \in [0,1]$, $P_i[t] = Q_i[t]$. Then, for each $\tau < t$, $P_i[\tau] = Q_i[\tau]$. In particular, $P[t] = Q[t]$ implies that for each $\tau \in [0,t]$, $P[\tau] = Q[\tau]$.

Lemma 2 is a straightforward generalization of the fact that, given $P \in \mathcal{P}$, the corresponding consumption process is uniquely determined. It states that, given a partial assignment $P[t]$, the corresponding part of consumption process leading to this partial assignment, i.e. $P[\tau]$, $\tau \in [0,t]$, is uniquely determined.

Let $R \in \mathcal{R}^N$ and $P \in \mathcal{P}$. Define $t(R)$ to be the largest $\tau \in [0,1]$ such that $P[\tau] = S(R)[\tau]$.[14] We say that the consumption processes representing P and $S(R)$ "diverge" at time $t(R)$. Let us call $t(R)$ the "divergence time" between P and $S(R)$.

Suppose that $P \neq S(R)$ and thus, $t(R) < 1$. By Lemma 2, for each $\tau \in [0,t(R)]$, $P[\tau] = S(R)[\tau]$ and for each $\tau > t(R)$, there is $i \in N$ such that $P_i[\tau] \neq S_i(R)[\tau]$. Note that the sets of objects that are still available at $t(R)$ are the same in both consumption processes. In the consumption process representing $S(R)$, agent i switches at $t(R)$ to an object, say a, which he prefers to each other available object. In the consumption process representing P, however, he switches to an object other than a (thus, less desirable than a). That is, he consumes an object that is less desirable than a before a is exhausted. We say that such **an agent *i* does not do his best on *a* at *t***. We state this observation as the following Lemma.

Lemma 3 Let $R \in \mathcal{R}^N$. The serial assignment matrix $S(R)$ is the only assignment matrix such that, for each $a \in A$, up to the time at which a is exhausted in the consumption process, each agent consumes objects that he finds at least as desirable as a.

Given a rule φ, define $t_\varphi(R)$ to be the divergence time between $\varphi(R)$ and $S(R)$.

Lemma 3 allows us to immediately obtain a characterization of the serial rule, proposed by Kesten et al. (2010), by means of the following axiom.

Ordinal fairness: For each $R \in \mathcal{R}^N$, each pair $i, j \in N$, and each $a \in A$ with $\varphi_{ja}(R) > 0$, we have $\sum_{b \in U(R_j,a)} \varphi_{jb}(R) \leq \sum_{b \in U(R_i,a)} \varphi_{ib}(R)$.

Theorem 1 (Kesten et al. (2010)) The serial rule is the only rule satisfying *ordinal fairness.*

Proof Let φ be an *ordinally fair* rule failing the premises of Lemma 3. This means that there is an object, say a, which is exhausted at some $t \in (0,1]$, but

[14] Continuity of the consumption processes, together with Lemma 2, implies that $\{\tau \in [0,1] : P[\tau] = S(R)[\tau]\}$ is a compact interval $[0,t(R)] \subseteq [0,1]$.

an agent, say i, starts consuming an object he ranks below a before t. Let j be an agent who ends consuming a positive share of a at t. By Lemma 1, $\sum_{b\in U(R_j,a)} \varphi_{jb}(R) = t$ and $\sum_{b\in U(R_i,a)} \varphi_{ib}(R) < t$, a violation of *ordinal fairness*.[15]
$\square$

3 Main Result

Our main result is a characterization of the serial rule.

Theorem 2 The serial rule is the only rule satisfying *sd efficiency, sd no-envy,* and *bounded invariance.*

Proof The serial rule satisfies *sd efficiency* and *sd no-envy* (Bogomolnaia and Moulin, 2001). To check *bounded invariance*, consider the following:

Claim 1. Let $R \in \mathcal{R}^N$ and $t \in [0,1]$. Let $i \in N$ and $d \in A$ be such that for each object o ranked below d, $(S(R)[t])_{io} = 0$. Let $R'_i \in \mathcal{R}$ be such that $R_i(d) = R'_i(d)$. Then, for each $\tau \in [0,t]$, $S(R'_i, R_{-i})[\tau] = S(R)[\tau]$.

Claim 1 follows from the definition of the serial rule. Indeed, for each time up to t, each agent consumes his best available object, which remains the same at both R and (R'_i, R_{-i}); before time t, agent i can still consume objects from $U(R_i, d)$, and nothing changes for the other agents. *Bounded invariance* readily follows from Claim 1, by choosing t as the time at which agent i ends consuming a in the premise of *bounded invariance.*[16]

Conversely, let φ be a rule satisfying these axioms. Suppose, by contradiction, that $\varphi \neq S$. Then, for some $R \in \mathcal{R}^N$, $t_\varphi(R) < 1$. Let $R^* \in \mathcal{R}^N$ be such that $t \equiv t_\varphi(R^*)$ is as small as possible: $t = t_\varphi(R^*) = \min_{R\in\mathcal{R}^n} t_\varphi(R)$. Since $\varphi \neq S$, $t < 1$.

There exists $i \in N$ not doing his best on an object, say a at t. Let a be agent i's most preferred object among the ones that are still available at t. Since agent i never returns to object a after time t, $\varphi_{ia}(R^*)=(\varphi(R^*)[t])_{ia}$. By *sd efficiency*, there is $j \in N \setminus \{i\}$ who consumes a positive share of a at some $t' > t$. (Thus, $\varphi_{ja}(R^*) > 0$.)

[15] Since we assume that $|A| = |N|$, *ordinal fairness* itself characterizes the serial rule. Suppose that for each object a, there are q_a copies of a and $\sum_{a\in A} q_a \geq |N|$ as in Kesten et al. (2010). Then, ordinal fairness, together with the following axiom, characterizes the same rule.

Non-wastefulness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $b \in A$ with $\varphi_{ib}(R) > 0$, if aR_ib, then $\sum_{k\in N} \varphi_{ka}(R) = q_a$.

The proof starts in the same way as above, but we also need to consider the case when object a is never exhausted. In this case, if some agent i ever consumes an object worse for him than a, *non-wastefulness* is violated.

[16] or as the earliest time at which agent i starts consuming a positive share of an object less preferred than a.

Let $A_j = U(R_j^*, a) \setminus U(R_i^*, a)$ and $A_{ij} = U(R_j^*, a) \cup U(R_i^*, a)$. By *sd efficiency*, for each $o \in A_j$, $\varphi_{io}(R^*) = 0$. Otherwise, agents i and j can improve their welfare by exchanging a fraction of these objects and a. Thus, $\sum_{o \in A_{ij}} \varphi_{io}(R^*) = t$.

Let $R'_i \in \mathcal{R}$ be such that $R_i^*(a) = R'_i(a)$ and the objects in A_j are pushed upward to be placed just below a. Let $R' = (R'_i, R^*_{-i})$.

Since t is chosen as the earliest divergence time between $\varphi(R)$ and $S(R)$ across all $R \in \mathcal{R}^N$, we have that for each $\tau \leq t$, $\varphi(R')[\tau] = S(R')[\tau]$. By Claim 1, $S(R')[\tau] = S(R^*)[\tau]$. Hence, $\varphi(R')[\tau] = \varphi(R^*)[\tau]$.

By *bounded invariance*, $\varphi_{ja}(R') = \varphi_{ja}(R^*) > 0$. Recall that in the consumption process representing $\varphi(R^*)$, agent j acquires a positive share of a at some $t' > t$. Since $\varphi_j(R')[t] = \varphi_j(R^*)[t]$, in the consumption process representing $\varphi(R')$, agent j also acquires a positive share of a at some $t'' > t$. By Lemma 1, agent j's total share of A_{ij} at $\varphi(R')$ is at least $t'' > t$.

Since $\varphi_{ja}(R') = \varphi_{ja}(R^*) > 0$, *sd efficiency* implies that for each $o \in A_j$, $\varphi_{io}(R') = 0$. By *bounded invariance*, for each $o \in U(R_i^*, a)$, $\varphi_{io}(R') = \varphi_{io}(R^*)$. Hence, agent i's total share of A_{ij} at $\varphi(R')$ is t. Recall that A_{ij} constitutes the "top" part of R'_i. However, his total share of A_{ij}, t, is smaller than the total share of A_{ij} that agent j receives, t'', a contradiction to *sd no-envy*. □

Remark We check that our axioms are independent. Let $N = \{1, 2, \cdots, n\}$ with $n \geq 3$ and $A = \{a, b, c, \cdots, z\}$ with $|A| = n$. Let $R \in \mathcal{R}^N$ be such that (i) $a \; R_1 \; b \; R_1 \; c \; R_1 \; \cdots$, $a \; R_2 \; c \; R_2 \; b \; R_2 \; \cdots$, $b \; R_3 \; c \; R_3 \; a \; R_3 \; \cdots$, and (ii) the most preferred object of each agent in $N \setminus \{1, 2, 3\}$ is in $A \setminus \{a, b, c\}$ and is different from that of each other agent in $N \setminus \{1, 2, 3\}$. Let $P \in \mathcal{P}$ be such that $P_1 = (1/2, 1/6, 1/3, 0, \cdots, 0)$, $P_2 = (1/2, 0, 1/2, 0, \cdots, 0)$, $P_3 = (0, 5/6, 1/6, 0, \cdots, 0)$, and each other agent is assigned probability 1 of receiving his most preferred object. Let ψ be a rule such that $\psi(R) = P$ and for each $R' \neq R$, $\psi(R') = S(R')$. This rule satisfies all of the axioms of Theorem 2 except for *bounded invariance*. The equal division rule, assigning $1/n$ of each object to each agent, satisfies all of the axioms except for *sd efficiency*. The sequential priority rule assigning to agent 1 his best object, to agent 2 his best among remaining objects, and so on, satisfies all of the axioms except for *sd no-envy*.

3.1 A Tighter Result

Our characterization still holds under weaker versions of *sd efficiency* and/or *bounded invariance*. First, a weaker notion of efficiency says that (i) no pair of agents can ever mutually improve their welfare by swapping a fraction of their assignments and (ii) there is no waste of an object that an agent prefers to some object in the support of his assignment.

Weak sd efficiency[17]: For each $R \in \mathcal{R}^N$, there is no $P \in \mathcal{P} \setminus \{\varphi(R)\}$ such that for each $i \in N$, $P_i \; R_i^{sd} \; \varphi_i(R)$ and $|\{i \in N : P_i \neq \varphi_i(R)\}| \leq 2$.

Next, we weaken *bounded invariance*. Let $P \in \mathcal{P}$, $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$. Let $a \in A$ be such that $p_{ia} = 0$. We say that R'_i is a **weak monotonic transformation of $\boldsymbol{R_i}$ at $\boldsymbol{(a, P)}$** if (i) $R_i|_{A\setminus\{a\}} = R'_i|_{A\setminus\{a\}}$ and (ii) $U(R_i, a) \supset U(R'_i, a)$. That is, R'_i is obtained from R_i by moving object a, of which agent i receives zero probability in P, upward in agent i's ranking.

Weak Monotonicity: Let $R \in \mathcal{R}^N$, $i \in N$, $a \in A$, and $R'_i \in \mathcal{R}$ be a weak monotonic transformation of R_i at $(a, \varphi(R))$. For each $k \neq i$ and each $b \in U^0(R'_i, a)$, $\varphi_{kb}(R'_i, R_{-i}) \geq \varphi_{kb}(R)$.

Proposition The serial rule is the only rule satisfying *weak sd efficiency*, *sd no-envy*, and *weak monotonicity*.[18]

Proof All arguments in the proof of Theorem 2 apply except for a small change: weak monotonicity now implies $\varphi_{ja}(R') \geq \varphi_{ja}(R^*) > 0$ and $\varphi_{ia}(R') \leq \varphi_{ia}(R^*)$. □

4 Unification of Known Results

We conclude by discussing possible generalizations of our model, as well as several related characterization results.

The first generalization is to introduce (finitely) multiple copies of each object.[19] The feasibility condition on assignment matrices now requires that the sum of the probabilities of each object assigned to the agents is AT most as large as the number of copies of this object. Our main result (Theorem 2) applies directly to this generalized model.

The second one is to introduce the null object, i.e. an option to receive no object.[20] Denote the null object by $\emptyset$. We call an object "acceptable" to an agent if he prefers the object to the null object, and "unacceptable" otherwise. All the

[17]It is a restatement of "2-ordinal efficiency" axiom in Hashimoto and Hirata (2011).

[18]Theorem 2 in Hashimoto and Hirata (2011) tightens their Theorem 1 by means of *weak sd efficiency* and a weaker notion than *sd no-envy*. In the version of the model that includes the null object, which we discuss in Section 4, our proof follows by imposing their weaker fairness condition.

[19]Moreover, we relax the assumption $|A| = |N|$ to be such that the total number of objects' copies is at least as large as the number of agents.

[20]This variation can be thought of as a subclass of the setting with multiple copies of some objects. Specifically, this is a subclass where at least one object "$\emptyset$" exists in $|N|$ copies. Hence, under any minimal efficiency requirement, each agent never receives a positive probability of each object that is less preferred than $\emptyset$.

arguments in our main result (Theorem 2) carry over, if only we set the number of copies of the null object to be $|N|$.

We now can unify several related characterization results. In Section 2.2, we already presented an alternative proof of Theorem 1 in Kesten et al. (2010) by using our proof technique. "Upper invariance" in Theorem 2 in Kesten at al. (2010) immediately implies *bounded invariance*: it requires the assignment of an object to remain unchanged (as in *bounded invariance*) under a larger class of changes in a single agent's preferences. In the presence of *sd efficiency*, "truncation robustness"[21] from Theorems 1, 2 in Hashimoto and Hirata (2011) is equivalent to *bounded invariance* on the class of models with null object which they consider.[22,23] Thus, these characterizations follow from ours. Lastly, Theorem 3 in Hashimoto and Hirata (2011), though the axioms they impose are different in spirit from ours, can still be easily obtained using our technique (see Appendix for a very short alternative proof).

Yet another generalization is to accommodate the possibility that each agent receives more than one object. If agents receive the same number of objects, then all of the axioms listed in Theorem 2 are still meaningful. Otherwise, however, it becomes difficult to compare agents' assignments directly, since the sum of probabilities assigned to each agent may differ. Heo (2010) introduces the notion of "normalized" *sd no-envy* to handle this problem: first normalize each agent's assignment by the number of objects that he has to receive, and then compare his normalized assignment to that of each other agent. The serial rule can also be generalized to this setting (Heo, 2010). All the arguments in Theorem 2 carry over with a slight adaptation: the "generalized serial rule" is the only rule satisfying *sd efficiency*, *normalized sd no-envy*, and *bounded invariance*.

[21] *Truncation robustness* requires that, whenever an agent changes his preferences so that each object below an object, say a, in his ranking becomes unacceptable, each agent's assignment of a should remain the same.

[22] *Bounded invariance*, introduced in our initial setting, is well-defined with no modification in this generalized setting. However, *truncation robustness* is silent in our initial model in which the null object is always ranked last. This axiom is meaningful only in the richer environment in which the null object can be ranked anywhere. This is also true for the other critical axioms in Hashimoto and Hirata (2011), the "Rawlsian criterion" and "independence of unassigned objects" (we discuss these axioms in Appendix).

[23] It is obvious that *bounded invariance* implies *truncation robustness*. Conversely, we show that *truncation robustness* implies *bounded invariance*. Given R_{-i}, let $R_i, R'_i \in \mathcal{R}$ be such that for an acceptable object at R_i and R'_i, say a, $R_i(a) = R'_i(a)$. Let R^*_i be such that $R^*_i(a) = R_i(a)$ and all objects below a are unacceptable at R^*_i . By *truncation robustness*, each agent's assignment of a remains the same at (R_i, R_{-i}) and (R^*_i, R_{-i}). Similarly, each agent's assignment of a remains the same at (R'_i, R_{-i}) and (R^*_i, R_{-i}). If a is unacceptable, then let $b \in A$ be the worst acceptable object for agent i. *Sd-efficiency* guarantees that i receives zero probability of each object below b at both R and (R'_i, R_{-i}). We can then apply the same argument as above to b instead of a.

Appendix: an alternative proof

We present here an alternative proof of Theorem 3 in Hashimoto and Hirata (2011), using consumption processes. Let $R_i \in \mathcal{R}$ and $a \in A$. We say that $R'_i \in \mathcal{R}$ is obtained from R_i by "sinking" a, if $\emptyset \; R'_i \; a$ and $R_i|_{A\setminus\{a\}} = R'_i|_{A\setminus\{a\}}$. We call $a \in A$ a **minimally preferred object at $\boldsymbol{R}$** if a is acceptable for at least one agent, and, for each agent, a is either unacceptable, or the worst among acceptable objects. Hashimoto and Hirata (2011) introduce two axioms:

Independence of unassigned objects:[24] Let $R \in \mathcal{R}^N$, $i \in N$, and $a \in A$ be such that $\varphi_{ia}(R) = 0$. If $R'_i \in \mathcal{R}$ is obtained from R_i by sinking a, then $\varphi(R) = \varphi(R'_i, R_{-i})$.

Rawlsian criterion:[25] Let $R \in \mathcal{R}^N$ and let a be a minimally preferred object at R. Let $R' \in \mathcal{R}^N$ be such that for each $j \in N$, R'_j is obtained from R_j by sinking a. Then,

(i) for each $b \in A \setminus \{a\}$, $\varphi^b(R) = \varphi^b(R')$, and
(ii) for each $P \in \mathcal{P}$ such that for each $b \in A \setminus \{a\}$, $P^b = \varphi^b(R')$,

$$\min_{j \in N:\, a R_j \emptyset} \sum_{o \in U^0(R_j, \emptyset)} p_{jo} \leq \min_{j \in N:\, a R_j \emptyset} \sum_{o \in U^0(R_j, \emptyset)} \varphi_{jo}(R).$$

Theorem 3 (**Hashimoto and Hirata (2011)**) The serial rule is the only rule satisfying *sd efficiency*, *independence of unassigned objects*, and the *Rawlsian criterion*.

Proof We omit showing that the serial rule satisfies these axioms. Conversely, let φ be a rule satisfying them. Suppose, by contradiction, that $\varphi \neq S$. For each $R \in \mathcal{R}^N$, let $M(R) \equiv \sum_{i \in N} |\{o \in A : o \; R_i \; \emptyset\}|$. Among all $R \in \mathcal{R}^N$ such that $\varphi(R) \neq S(R)$, choose R^* so as to minimize $M(R)$, i.e., $R^* = \arg \min_{R \in \mathcal{R}^N} M(R)$. Let $t = t_\varphi(R^*)$. Since $\varphi \neq S$, $t < 1$. Then, there is $i \in N$ not doing his best on an object, say a, at t. Let a be agent i's most preferred object among the ones that are still available at t. By *sd efficiency*, there is $j \in N \setminus \{i\}$ consuming a positive share of a at some $t' > t$. (Thus, $\varphi_{ja}(R^*) > 0$.)

[24] *Independence of unassigned objects* differs from *bounded invariance* in two respects. First, its premise is a change of preferences over some "dummy" objects (i.e., objects assigned zero probability). Second, on the conclusion side, *independence of unassigned objects* restricts the *whole* assignment matrix.

[25] The *Rawlsian criterion* is a fairness requirement based on the notion that "the welfare of an agent is measured by the probability that she receives an acceptable object" as Hashimoto and Hirata (2011) argue. Moreover, it encompasses an idea of "separability": each minimally preferred object should be assigned independently of the other acceptable objects. In spirit, this axiom is different from that underlying *sd no-envy*.

Claim 2. For each $(k,b) \in N \times A$ such that $(k,b) \neq (i,a)$, $\varphi_{kb}(R^*) > 0$.

Indeed, suppose that there is $(k,b) \neq (i,a)$ such that $\varphi_{kb}(R^*) = 0$. Let $R'_k \in \mathcal{R}$ be obtained from R^*_k by sinking b and let $R' = (R'_k, R^*_{-k})$. By *independence of unassigned objects*, $\varphi(R') = \varphi(R^*)$, and agent i still does not do his best on a at t, that is, $\varphi(R') \neq S(R')$. But this is impossible, since $M(R') < M(R^*)$.

Claim 3. There are no minimally preferred objects at the profile R^*.

Suppose by contradiction that b is a minimally preferred object at R^*. Let $R' \in \mathcal{R}^N$ be such that for each $k \in N$, R'_k is obtained from R^*_k by sinking b. The *Rawlsian criterion* gives, for each $o \neq b$, $\varphi^o(R^*) = \varphi^o(R')$. If $b \neq a$, agent i still does not do his best on a at t in the consumption process representing $\varphi(R')$. That is, $\varphi(R') \neq S(R')$, which contradicts $M(R') < M(R^*)$. If $b = a$, a positive share of $\varphi_{ja}(R^*)$ can be transferred to agent i,[26] in violation of condition (ii) of the *Rawlsian criterion*.

Now, for each $k \in N$, let $l(k)$ be his worst acceptable object at R^*_k. By Claim 3, for each agent k, there is another agent m such that $l(k) R^*_m l(m)$. Hence, we can find a "cycle" of agents $(k_1, \cdots, k_r)$, all distinct except that $k_1 = k_r$, and such that for each $2 \leq s \leq r$, $l(k_{s-1}) R^*_{k_s} l(k_s)$. Suppose that for each $s \in \{1, \cdots, r\}$, $(k_s, l(k_s)) \neq (i,a)$. Then, by Claim 2, for each $s \in \{1, \cdots, r\}$, $\varphi_{k_s l(k_s)}(R^*) > 0$, and we obtain a violation of *sd efficiency*. Thus, for some $s \in \{1, \cdots, r\}$, $(k_s, l(k_s)) = (i,a)$ and $\varphi_{ia}(R^*) = 0$. Without loss of generality, let $i = k_r$. Recall that $\varphi(R^*)[t] = S(R^*)[t]$ and a is the most preferred object for agent $k_r (= i)$ among the ones that are still available at time t. Since $l(k_{r-1})\ R_{k_r}\ l(k_r) (= a)$, object $l(k_{r-1})$ is exhausted no later than t.

Next, $l(k_{r-2})\ R_{k_{r-1}}\ l(k_{r-1})$ and, by Claim 2, $\varphi_{k_{r-1} l(k_{r-1})}(R^*) > 0$. Object $l(k_{r-1})$ is exhausted at t, and for each $\tau \leq t$, the partial assignment $\varphi(R^*)[\tau]$ is the same as that of the serial rule. Hence, $l(k_{r-2})$ is exhausted earlier than $l(k_{r-1})$, that is, even earlier than time t (in the consumption process representing the serial assignment, agent k_{r-1} does not start consuming $l(k_{r-1})$ before $l(k_{r-2})$ is exhausted). For the same reason, $l(k_{r-3})$ is exhausted earlier than $l(k_{r-2})$, $l(k_{r-4})$ is exhausted earlier than $l(k_{r-3})$, etc., and finally, $l(k_1) (= a)$ is exhausted earlier than $l(k_2)$, or even earlier than t. But this contradicts the fact that agent i does not do his best on a at t in the consumption process representing $\varphi(R^*)$. □

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