The Home Market Effect in a Multicountry Space

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Abstract

The concept of home market effect (HME) refers to the phenomenon that firms agglomerate more than the market share in a large country. Krugman (1980) set up two models, showing that firms agglomerate more than the market share and that the wage is higher in the larger country. In Takahashi et al (2010), the equivalence between a larger firm share and a higher wage is proved. All of them are based on a space of two countries. This paper examines the HME in a multicountry space. We find that both the wage and firm shares are ordered according to the countries’ sizes. And, they exhibit U-shape relationship with respect to the falling transport costs.

1 Introduction

The world economy has been reshaped greatly in recent decades, by the formation of the North American Free Trade Agreement (NAFTA), the European Union (EU), and the Association of Southeast Asian Nations plus China, Japan and Korea (ASEAN Plus Three). Those communities decrease the tariffs and increase the competing forces among national economies, so many politicians and economists are eager to know whether their countries will benefit from such a trade globalization or not. For example, the Japanese commitment to the Trans Pacific Partnership (TPP) has raised an earnest discussion by both ruling and opposition parties over trade liberalization.

Trade theory is developed to answer such problems. The traditional trade theory explains the trade as a result of comparative advantage among the related countries.
For example, Ricardo (1817) shows the importance of heterogeneous production technologies, while Heckscher (1919) and Ohlin (1933) reveal the role of production resource in international trade. The comparative advantage theory is very useful to explore the inter-industry trade between a developed country and a developing country. However, it has mainly two problems. First, it cannot explain the intra-industry trade, involving the import and export of similar goods. Importantly, the intra-industry share of manufacturing trade has increased significantly since the late 1980s across many OECD countries. Second, as criticized by Deardorff (1984, P.470), transport costs are almost universally ignored in traditional trade models in the sanguine hope that if included they would not materially affect the results. Such an omission makes it impossible to consider the effect of decreasing trade costs.

In recent three decades, new trade theory (NTT) is developed to overcome the two problems. NTT focuses on the technology of increasing return to scale and the consumer preference for diversity. A seminal paper of Krugman (1980) establishes two general equilibrium models successfully incorporating monopolistic competition and trade costs. An important result is that a country which has a relatively larger market size tends to attract a more-than-proportionate share of firms. This is called the home market effect (HME) in the literature.

More specifically, Section 3 of Krugman (1980) assumes the economy space as a world of two countries. There are two symmetric industries in the manufacturing sector, and two groups of consumers in two countries, each consumes the differentiated goods in one industry only. The country sizes are equal, but each has a larger share of one consumer group. Krugman shows that a more-than-proportionate share of firms agglomerate in the country with a larger consumer group in the industry. This model heavily depends on the symmetry between industries and countries, so the wages in two countries are equalized. This model is reformulated by Helpman and Krugman (1985), who add an agricultural sector. The homogeneous agricultural good is produced under a technology of constant returns to scale. Labor in the only production factor, and the wages in two countries are equalized by assuming free transportation of the agricultural good. This assumption is called the condition of factor price equalization in Behrens et al (2009).

Martin and Rogers (1995) introduce another production factor, mobile capital, into the NTT research. Capital is assumed to be owned averagely by all residents, and the capital rents are remitted to its country of origin. Therefore, such a framework is called footloose capital model in the literature. In the manufacturing production, mobile capital is input as the fixed cost while immobile labor is input as the marginal cost. This model
also contains an agricultural sector under the condition of factor price equalization.

However, the wages in different countries are never equalized in the real world. Free trade of the homogeneous good is criticized by some researchers (e.g., Davis (1998)) and the agricultural good is argued as an outside good by Crozet and Trionfetti (2008).

Interestingly, Section 2 of Krugman (1980) finds that a higher wage in a larger country is another feature of the home market effect. There, the economic space is assumed to be a world of two countries which are different in size with one manufacturing sector producing differentiated goods. Labor is again the only input to produce the goods. In that model, the wage in the larger country is higher than the wage in the smaller country. Meanwhile, the firms shares are proportional, because of the trade balance in the only manufacturing sector.

A recent paper of Takahashi et al (2010) successfully discloses these two features (i.e., a higher wage and a large firm share in the larger country) in one model. By deleting the agricultural sector in the footloose capital model, the wages are allowed to be different between two countries. More interesting, he finds that both the wage and the firm share exhibit as U-shaped curves with respect to falling trade costs. This result is consistent with recent studies by models with positive agricultural transport costs in Zeng and Kikuchi (2009), Takatsuka and Zeng (2009).

Most papers in international trade assume two countries. In a space of two countries, there is only one way in which countries can interact. Moving away from one country automatically implies that firms go to the other country. Whereas in the case of three countries, there are two ways in which these countries can interact. In other words, we need a multicountry model to incorporate complex feedbacks. Behrens et al (2009) extend the model of Helpman and Krugman (1985) to a setup with an arbitrary number of countries. They keep the agricultural sector but assume a Ricardian comparative advantage in its production. Although the wages are, therefore, different across countries, the differentials are given exogenously and fixed, which is again far from the real world. In fact, the wage rates in the Newly Industrializing Economies (NIEs) and BRICs increase rapidly in recent years, shrinking their distances to developed countries.

The aim of this paper is to analyze the HME and the wage in a multicountry space. Without the agricultural sector, we extend the model of Takahashi et al (2010), and show that similar results hold in the case of three countries. This model also exhibits the relationship between the HME and the wage differential. The order of the wages and the order of firm shares are both determined by the country sizes.

The rest part of this paper is constructed as follows. In Section 2, we establish the
model of \( n \) countries. Section 3 provides the equilibrium result for the case of symmetric trade costs. Section 4 summarizes the conclusions.

2 The model

The global economy consists of \( n \) countries which are called country \( i, (i = 1, 2, \ldots, n) \). The countries have the same conditions, except their population sizes. There are \( L \) units of labor and \( K \) units of capital totally. To rule out the Heckscher-Ohlin comparative advantage, each resident is assumed to hold the same amount of capital. Therefore, the capital owner share is the same as the labor share in each country. We denote them as \( \theta_i, i = 1, 2, 3 \). Without loss of generality, we assume that \( \theta_1 \geq \theta_2 \geq \ldots \geq \theta_n \).

In this paper, we assume only one (manufacturing) sector, whose production is under the technology of increasing returns to scale. We don’t need a sector of constant returns to scale, which is called the agricultural sector in the literature. So, the consumers gain the utility only from manufacturing goods, described by utility function \( U = M \), where \( M \) represents the consumption of a composite of manufactured varieties. There is a continuum of varieties. The consumption in country \( i \) is

\[
M_i = \left[ \int_{0}^{N} d_i(n)^{\frac{\sigma-1}{\sigma}}dn \right]^{\frac{1}{\sigma-1}},
\]

where \( \sigma \geq 2 \) represents the elasticity of substitution between two manufactured varieties\(^1\), \( N \) is the number of varieties, and \( d_i(n) \) is the demand of a typical manufactured good \( n \) in country \( i \). The varieties are supposed to be symmetric, so we can omit the variety name and simply use \( d_n \) to indicate the demand of each variety.

Maximization of consumers’ utility function derives the following demand functions

\[
d_{ij} = \frac{p_{ij}^\sigma}{P_j^{1-\sigma}} Y_j
\]

where \( d_{ij} \) is the demand of a variety made in country \( i \) and sold in country \( j \), \( p_{ij} \) is its price, \( Y_i \) is the national income in country \( i \), \( P_i \) is the manufacturing price index in country \( i \) given by

\[
P_i = \left[ \int_{0}^{N} p_i(n)^{1-\sigma}dn \right]^{\frac{1}{1-\sigma}}.
\]

\(^1\)While most theoretical papers in this field assumes \( \sigma > 1 \), some empirical studies reveal that \( \sigma > 2 \). For example, Redding and Venables (2004) and Hanson (2005) find that \( \sigma \) is between 5 and 10. Therefore, we assume \( \sigma \geq 2 \) in this paper, which simplifies our later analysis a lot.
Next we turn to the production side of the economy. We assume the same technology in all countries. Fixed input of one unit of capital and a marginal input of $(\sigma - 1)/\sigma$ units of labor are required in production. The firm share in each country is equal to the capital share $\lambda_i$. Then the number of firms in $i$ is $\lambda_i K$.

International shipment of any variety incurs costs, which evidently depend on the geographical distance between the origin and destination countries. As shown in Behrens et al (2009), the geographical feature of countries might produce hub effect have great impact on trade patterns. Since the purpose of this paper is to clarify the trade pattern formed by the so-called second nature (increasing return to scale technology, monopolistic competition, trade costs and etc.), we rule out the first nature difference across countries by assuming that the transport costs are the same for all pairs of countries. Specifically, $\tau \geq 1$ units of a manufactured good must be shipped for one unit between any two countries. Then we have $p_{ij} = \tau_{ij} p_i$ for any $i, j = 1, \ldots, n$ where

$$
\tau_{ij} = \begin{cases} 
\tau & \text{if } i \neq j \\
1 & \text{if } i = j.
\end{cases}
$$

A firm located in country $i$ sets its price to maximize its profit

$$
\pi_i = \sum_{j=1}^{n} \left( p_{ij} d_{ij} - \frac{\sigma - 1}{\sigma} w_i d_{ij} \right) - r_i
$$

where $r_i$ is the capital rent in country $i$. The first-order condition to maximize (2) gives prices

$$
p_{ij} = w_i \tau_{ij}
$$

Therefore, the price indices are simplified as

$$
P_i = \left( \sum_j K \lambda_j w_j^{1-\sigma} \phi_{ij} \right)^{\frac{1}{1-\sigma}},
$$

where $\phi_{ij} \equiv \tau_{ij}^{1-\sigma} \in [0, 1]$ is the trade freeness between countries $i$ and $j$.

Because of the free-entry condition of firms, the profit of firms is zero in equilibrium. Then, we get the capital rent as

$$
r_i = \frac{1}{\sigma K} \left( \sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_k w_k^{1-\sigma} \phi_{ik} Y_j \right).
$$

Next, the national incomes are the sum of the capital rent and wages,

$$
Y_i = \theta_i K \sum_j \lambda_j r_j + w_i \theta_i L.
$$

Finally, we choose the wage in country 1 as numéraire so that $w_1 = 1$. 

5
3 Equilibrium

Firms choose the country providing higher capital returns. Following the literature, we apply a standard dynamic system to describe the international capital flows:

$$\frac{d\lambda_i}{dt} = \sum_{j=1}^{n} \lambda_i \lambda_j (r_i - r_j).$$

Intuitively, capital moves from country $j$ to country $i$ in proportion to the present shares $\lambda_i$ and $\lambda_j$ in the countries and the rent differential.

In an interior equilibrium, firms are located in all countries. Then the capital rents are equal in all countries: $r_1 = \cdots = r_n$. By equation (4), the capital rents satisfy

$$r_i = \frac{\sum_j Y_j}{\sigma K} = r \quad \text{for all} \quad i = 1, \cdots, n. \quad (6)$$

Substituting (6) to (5), we obtain

$$Y_i = \frac{L \theta_i}{\sigma - 1} \left[ \sum_{j=1}^{n} \theta_j w_j + w_i (\sigma - 1) \right], \quad (7)$$

yielding

$$r = \frac{L}{K(\sigma - 1)} \sum_j \theta_j w_j.$$

Next we turn to the labor market. The total labor costs of firms in each country are

$$l_i = \lambda_i \frac{\sigma - 1}{\sigma} \sum_{j=1}^{n} Y_j = L \lambda_i \sum_{j=1}^{n} \theta_j w_j.$$ \hspace{1cm}

Meanwhile, the labor supplies in each country are $\theta_i L$, respectively. Accordingly,

$$L \lambda_i \sum_{j=1}^{n} \theta_j w_j = w_i \theta_i L,$$

hold, which derive a relationship between wages and firm shares:

$$\lambda_i = \frac{w_i \theta_i}{\sum_{j=1}^{n} \theta_j w_j}. \quad (8)$$

According to (8), $\lambda_i$ satisfies $0 < \lambda_i < 1$, since the wages are all positive. Therefore, our model has interior equilibrium only.

The wages are determined endogenously by the market-clearing conditions of labor market in each country:

$$\lambda_i K \frac{\sigma - 1}{\sigma} \sum_j \tau_{ij} d_{ij} = L \theta_i.$$
By use of (1), (3), (7), (8), the above expressions can be written as:

\[ 1 - \frac{w_i^{1-\sigma}}{1 - \phi} \sum_{j=1}^{n} H_j(w) \phi_{ij} = 0, \quad i = 1, \ldots, n, \]  

(9)

where

\[ H_j(w) \equiv \frac{(1 - \phi) \theta_j}{\sigma \sum_{k=1}^{n} \theta_k w_k^{2-\sigma} \phi_k}, \quad j = 1, \ldots, n. \]  

(10)

On the other hand, we can take (9) as equations for \( H_1(w), \ldots, H_n(w) \). Solving them obtains

\[ H_j(w) = w_j^{\sigma-1} - \frac{\phi}{1 + (n-1)\phi} \sum_{k=1}^{n} w_k^{\sigma-1}, \quad j = 1, \ldots, n. \]  

(11)

Two expressions of (10) and (11) for \( H_j(w) \) form the following equations for the wage rates \( w_1, w_2, \ldots, w_n \):

\[ \frac{\sigma - 1}{\sigma} w_j + \frac{1}{\sigma} \sum_{k=1}^{n} \theta_k w_k \frac{w_j^{2-\sigma} + \phi \sum_{k=1}^{n} \theta_k w_k^{2-\sigma}}{1 - \phi \sum_{k=1}^{n} \theta_k^{2-\sigma}} + \frac{\phi}{1 + (n-1)\phi} \sum_{k=1}^{n} w_k^{\sigma-1} = w_j^{\sigma-1}, \quad j = 1, \ldots, n. \]  

(12)

Note that one of the above equations is redundant, and \( n - 1 \) of them can determine \( w_2, \ldots, w_n \).

3.1 The home-market effect

Behrens et al (2009) first provide a definition for the Multiple-region HME. For countries with size order \( \theta_1 \geq \ldots \geq \theta_M \), the HME is defined as the phenomenon of

\[ \frac{\lambda_1}{\theta_1} \geq \ldots \geq \frac{\lambda_M}{\theta_M}. \]  

(13)

Stated differently, the order of firms share reflects the order of countries’ market size. This means the country of a larger market size always hold relatively larger firm share.

Meanwhile, there is an alternative definition of the HME in terms of wage that, other things equal, the wage is higher in a larger country (Krugman, 1991, p.491; Behrens et al, 2009, footnote 1). That is, \( 1 = w_1 \geq w_2 \geq \cdots w_n \) in our model. Fortunately, we have
Proposition 1 The two definitions of the HME are equivalent.

Proof: Expression (8) can be rewritten as

$$\frac{\lambda_i}{\theta_i} = \frac{w_i}{\sum_j \theta_j w_j}.$$ 

Therefore, (13) holds iff the wages are ordered as \(w_1 \geq w_2 \geq \ldots \geq w_n\).

The wage equations (12) are complicated that we have no analytical solution. To have an image of the wage relationship, we first perform numerical simulations to observe the tendency of the wages with respect to the trade freeness. Figures 3.1 shows the wages \(w_2\) and \(w_3\) when \(n = 3, \theta_1 = 0.5, \theta_2 = 0.3, \theta_3 = 0.2, \sigma = 3\).

![Figure 1: Wages \(w_2\) and \(w_3\)](image)

In Figure 3.1, a larger country provides a higher wage rate. Intuitively, a higher wage in a large country can keep full labor employment in all countries. If production cost of wages were the same in all countries, it would always be more profitable to produce in a larger country. The wage differential offsets this advantage. At the same time, in the country of a higher wage, each firm reduces its labor share in the total costs. The HME appears as a result of full employment.

In a larger country, two effects are observed. One is called the market size effect. Because consumers with higher incomes increase their demands for manufactured goods, more firms are likely to locate in the country. The other effect is called the production cost effect. The firms must pay higher wages to labor, so the firms are likely to escape from the country. We show that, as a balance of these two effects, the firms agglomerate in larger countries, and the wage rate is actually higher in a larger country.
Proposition 2 For any positive and finite transport cost, if $\theta_1 > \theta_2 > \cdots > \theta_n$, then the wages are ordered as $1 = w_1 > w_2 > \cdots > w_n$ and the HME occurs.

Proof: See Appendix B.

\[ \square \]

3.2 U-shape of wages

Figure 3.1 exhibits how the wages change with falling trade costs. They are U-shaped curves, first decreasing and then increasing in $\phi$. Because of the simple relationship between wage and industry share, we know that the firm share in each country takes an inverted U-shape. Both can be proven analytically.

Proposition 3 The firm share in each country $j = 1, \cdots, n$ evolves as an inverted U-shape curve while the wage in country $j = 2, \cdots, n$ evolves as a U-shaped curve with respect to trade freeness.

Proof: See Appendix C.

The (inverted) U shape of the firm shares indicates a re-dispersion process of economic activity, which is one of the typical results in NTT and NEG. This U-shaped pattern of location evolution is a sensitive issue because some politicians and economists of small countries worry about whether their countries could be de-industrialized permanently through increased economic integration. Several papers, like Krugman and Venables (1990, 1995), Venables (1996), Puga and Venables (1997), Puga (1999), and Zeng and Kikuchi (2009) reveal this fact theoretically, while Brülhart and Torstensson (1996) find some empirical support for this evolution pattern in EU.

3.3 Trade balance

We can calculate the balance-of-payment in three countries at equilibrium. First, the net flow of capital rent in country $i$ is

$$ B^r_i = (\theta_i - \lambda_i)K \sum_j \lambda_j r = \frac{L \theta_i}{\sigma - 1} \left[ \sum_j \theta_j w_j - w_i \right]. $$

And the trade surplus of manufactured goods in country $i$ is calculated as

$$ B^t_i = \lambda_i K \sum_j p_{ij} d_{ij} - K \sum_k \lambda_k p_{ki} d_{ki} $$

$$ = \frac{L \theta_i}{\sigma - 1} \left( w^2_i - \sigma \sum_j \theta_j \phi_{ij} \frac{\sum_k \theta_k w_k + (\sigma - 1)w_j}{\sum_k \theta_k w_k^{2-\sigma} \phi_{jk}} \right) - \frac{\theta_i L}{\sigma - 1} \left( \sum_j \theta_j w_j + w_i (\sigma - 1) \right). $$
Then, the total balance-of-payment in country $1$ is

$$B_i = B'_i + B''_i = -\frac{Lw_i\theta_i}{\sigma - 1}\sigma F_i = 0,$$

where the last equation is from (9). Other countries can be checked similarly. In summary, the balance-of-payment is ensured by the wage equations.

4 Conclusion

This paper analyzes the relationship between the country’s size and the wage in a multi-country world. The existing literature on a world of two countries predicts that the larger country has a higher firm share and a higher wage. We show that when the number of the countries is arbitrary, this prediction persists in the condition of symmetric transport costs. In particular, this model predicts the occurrence of the HME and the order of the wages turns to be the same as the order of countries’ sizes. And the wages differential causes two effects, the market size effect and the production cost effect, which cause the HME.

In conclusion, we find that the HME in terms of firms share and the HME in terms of the wage are equivalent regardless of the existence of an arbitrary country.

Appendix A: The proof of the existence

We prove that equations (12) have a solution $w^* \in [\phi^{\frac{1}{\sigma - 1}}, 1]^n$. For convenience, we rewrite (12) as

$$H_j(w) + \frac{\phi}{1 + (n - 1)\phi} \sum_{k=1}^{n} w_k^{\sigma - 1} = w_j^{\sigma - 1}, \quad \forall j = 1, \ldots, n. \tag{14}$$

Since $w_1 = 1$, equation (14) for $j = 1$ gives

$$\frac{\phi}{1 + (n - 1)\phi} \sum_{k=1}^{n} w_k^{\sigma - 1} = 1 - H_1(w).$$

Therefore, the remaining $n - 1$ equations of (14) can also be written as

$$\left(H_j(w) - H_1(w) + 1\right)^{\frac{1}{\sigma - 1}} = w_j, \quad \text{for} \quad i = 2, \ldots, n. \tag{15}$$

Lemma 1 It holds that $H_1(w) \leq 1 - \phi$ for $w \in [\phi^{\frac{1}{\sigma - 1}}, 1]^n$. 

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Proof: The following inequalities

\[
\frac{1}{\sigma} \left(1 - \frac{n}{\sum_{k=1}^{n} \theta_k w_k}\right) \geq 0 \geq \phi \left(-\frac{n}{\theta_1 w_1^2 - \sigma} \right)
\]

imply

\[
\frac{\sigma - 1}{\sigma} + \frac{1}{\sigma} \sum_{k=1}^{n} \theta_k w_k \leq 1 - \phi + \phi \sum_{k=1}^{n} \frac{\theta_k w_k^2 - \sigma}{\theta_1 w_1^2 - \sigma},
\]

which can be rewritten as \(H_1(w) \leq 1 - \phi\).

Lemma 2 Inequality \(H_j(w) \leq H_1(w)\) holds for all \(j = 2, \cdots, n\) and \(w \in [\phi \frac{1}{n}, 1]^n\).

Proof: Since \(\sigma \geq 2\), we have \(w_j^{2 - \sigma} \geq 1\) when \(w_j \leq 1\). Therefore, (10) implies \(H_j(w) \leq H_1(w)\) directly for \(w \in [\phi \frac{1}{n}, 1]^n\).

Since \(H_j(w) \geq 0\), Lemma 1 concludes that the LHS of (15) \(\geq \phi \frac{1}{n}\). Meanwhile, Lemma 2 says that the LHS of (15) \(\leq 1\). Therefore, (15) form a map of \(w\) from \([\phi \frac{1}{n}, 1]^n\) to \([\phi \frac{1}{n}, 1]^n\). The fixed point theorem tells us that equation (15) has a solution \(w^* \in [\phi \frac{1}{n}, 1]^n\).

Appendix B: The proof of the wage order

Note that \(w_1 = \cdots = w_n = 1\) are the only solution of (9) when \(\phi = 1\).

Lemma 3 For \(\phi \in [0,1)\) and different countries \(i,j\), \(w_i = w_j\) holds iff \(\phi = 0\).

Proof: Necessity. When \(w_i = w_j\), (15) implies \(H_i(w) = H_j(w)\), deriving

\[
\frac{\phi}{1 - \phi} \left(\frac{1}{\theta_i} - \frac{1}{\theta_j}\right) \sum_{k=1}^{n} \theta_k w_k^{2 - \sigma} = 0.
\]

Then \(\phi = 0\) because \(\theta_i \neq \theta_j\).

Sufficiency. When \(\phi = 0\), the definition (10) of \(H_j(w)\) gives

\[
H_i(w) = \frac{(\sigma - 1)w_i + \sum_{k=1}^{n} \theta_k w_k}{\sigma w_i^{2 - \sigma}},
\]

and (14) is simplified as

\[
\sum_{k=1}^{n} \theta_k w_k = w_j, \quad \text{for any country } j,
\]

implying \(w_i = w_j\).
Lemma 4 Wages $w_i(\phi)$ are ordered as $w_1 = 1 > w_2(\phi) > \cdots > w_n(\phi)$ for all $\phi \in (0, 1)$.

Proof: We rewrite (10) as

$$\left( w_j^{2-\sigma} + \frac{\phi}{1-\phi} \sum_{k=1}^{n} \frac{\theta_k w_k^{2-\sigma}}{\theta_j} \right) H_j(w) = \frac{\sigma - 1}{\sigma} w_j + \frac{1}{\sigma} \sum_{k=1}^{n} \theta_k w_k.$$ 

Taking the total differential of two sides with respect to $\phi$ at $\phi = 0$ (where $w_j = 1$ holds for all country $j$), we obtain:

$$\frac{dH_j(w)}{d\phi} \bigg|_{\phi=0} = \left( \sigma - 1 - \frac{1}{\sigma} \right) w_j'(0) + \frac{1}{\sigma} \sum_{k=1}^{n} \theta_k w_k'(0) - \frac{1}{\theta_j}, \quad \text{for} \quad j = 1, \cdots, n.$$ 

Recalling that $w_1(w) = 1$ holds for all $\phi$, the above equation for $j = 1$ implies

$$\frac{dH_1(w)}{d\phi} \bigg|_{\phi=0} = \frac{1}{\sigma} \sum_{k=1}^{n} \theta_k w_k'(0) - \frac{1}{\theta_1},$$

Now we take the total differential of (15) with respect to $\phi$ at $\phi = 0$, obtaining

$$w_j'(0) = \frac{1}{\sigma - 1} \left( \left. \frac{dH_j(w)}{d\phi} \right|_{\phi=0} - \left. \frac{dH_1(w)}{d\phi} \right|_{\phi=0} \right)$$

$$= \frac{1}{\sigma - 1} \left[ \left( \sigma - 1 - \frac{1}{\sigma} \right) w_j'(0) + \frac{1}{\theta_1} - \frac{1}{\theta_j} \right].$$

Therefore, we have

$$w_j'(0) = \sigma \left( \frac{1}{\theta_1} - \frac{1}{\theta_j} \right) \quad \text{for} \quad j = 2, \cdots, n,$$

so that $w_1'(0) > w_2'(0) > \cdots > w_n'(0)$. On the other hand, $w_i'(\phi)$ do not cross in $\phi \in (0, 1)$ according to Lemmas 3. Therefore, $1 > w_2(\phi) > \cdots > w_n(\phi)$ holds for all $\phi \in (0, 1)$. □

Appendix C: The proof of the U-Shape

We first show that $w_j'(1) > 0$ for all $j = 2, \cdots, n$. Taking the total differential of (15) with respect to $\phi$ and let $\phi \to 1$, we obtain

$$n(\sigma - 1)w_j'(1) = (\sigma - 1) \sum_{k=1}^{n} w_k'(1) - n\theta_j + 1 \quad \text{for all} \quad j = 1, \cdots, n.$$ 

Since $w_i'(1) = 0$ holds, the above equality for $j = 1$ leads to $(\sigma - 1) \sum_{k=1}^{n} w_k'(1) = n\theta_1 - 1$. Consequently,

$$n(\sigma - 1)w_j'(1) = n(\theta_1 - \theta_j) > 0$$

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holds for all \( j = 2, \ldots, n \), so that \( w_j'(1) = (\theta_1 - \theta_j)/(\sigma - 1) > 0 \).

Recall that we obtained \( w_j'(0) < 0 \) for \( j = 2, \ldots, n \) in Appendix B, we know that there exists a \( \phi^* \) satisfying \( w_i'\phi^* = 0 \). In the following, we show that the uniqueness of \( \phi^* \).

Rewrite (15) as

\[
I_j(w, \phi) \equiv \left( H_j(w) - H_1(w) + 1 \right)^{\frac{1}{\sigma - 1}} - w_j = 0, \quad \text{for } j = 2, \ldots, n.
\]

Then \( w_j'(\phi^*) = 0 \) holds if and only if

\[
I_j(w(\phi^*), \phi^*) = 0
\]

\[
\frac{\partial I_j(w, \phi)}{\partial \phi} \Bigg|_{\phi=\phi^*, w=w(\phi^*)} = 0
\]

Equation (17) immediately derives the following relationship:

\[
H_j(w(\phi^*)) = \frac{\theta_j}{\theta_1} \frac{w_j(\phi^*)^{2-\sigma} + \frac{\phi^*}{1-\phi^*} \sum_{k=1}^{n} \theta_k w_k(\phi^*)^{2-\sigma}}{1 + \frac{\phi^*}{1-\phi^*} \sum_{k=1}^{n} \theta_k w_k(\phi^*)^{2-\sigma}} H_1(w(\phi^*)�)
\]

According to (16), \( w(\phi^*) \) solves the following solution:

\[
J_j(w) = \frac{\theta_j w_j^{2-\sigma} - \theta_1}{\theta_1} \frac{H_1(w)}{1 + \frac{\phi^*}{1-\phi^*} \sum_{k=1}^{n} \theta_k w_k^{2-\sigma}} + 1 - w_j^{\sigma-1} = 0.
\]

Since \( 1 - w_j^{\sigma-1} \) is positive, we know that \( \theta_j w_j^{2-\sigma} < \theta_1 \). On the other hand, it holds that

\[
\frac{\partial J_j(w)}{\partial w_j} = \frac{1}{\theta_1} \frac{(2-\sigma)\theta_j w_j^{1-\sigma} \left[ 1 + \frac{\phi}{1-\phi} \sum_{k=1}^{n} \theta_k w_k^{2-\sigma} + \frac{\phi}{1-\phi} \frac{1}{\theta_1} (\theta_1 - \theta_j w_j^{2-\sigma}) \right] \frac{H_1(w)}{1 + \frac{\phi^*}{1-\phi^*} \sum_{k=1}^{n} \theta_k w_k^{2-\sigma}} + \frac{\theta_j w_j^{2-\sigma} - \theta_1}{\theta_1} - 1}{\theta_1 \left( 1 + \frac{\phi^*}{1-\phi^*} \sum_{k=1}^{n} \theta_k w_k^{2-\sigma} \right) \frac{\partial H_1(w)}{\partial w_j}}
\]

According to the facts of \( \theta_j w_j^{2-\sigma} < \theta_1 \), \( H_1(w) > 0 \) and \( \partial H_1(w)/\partial w_j > 0 \), we know that \( \frac{\partial J_j(w)}{\partial w_j} \) is negative. Meanwhile, we have

\[
J_j(1) = \frac{\theta_j - \theta_1}{\theta_1 + \frac{\phi^*}{1-\phi^*}} H_1(1) < 0,
\]

\[
J_j(0) = \frac{-1}{1 + \frac{\phi^*}{1-\phi^*} \theta_1} H_1(0) + 1 = \frac{-1}{1 + \frac{\phi^*}{1-\phi^*} \theta_1} \frac{1}{1 + \frac{\phi^*}{1-\phi^*}} + 1 > 0
\]

Therefore \( J_j(w) = 0 \) has only one solution, showing the uniqueness of \( w(\phi^*) \). Accordingly, solution \( \phi^* \) of (16) and (17) is unique.
References


